THE MATHEMATICAL GAZETTE

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THE
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THE FLEXAGON FAMILY

BY R. F. WHEELER

In Note 2449 (M.G. 1954, p. 213), Dr. F. G. Maunsell gave an account of two interesting figures, the flexagon (F), and a modified "hexahexaflexagram" (H). I have been prompted to investigate this subject further, and find that there is really an infinite family of such figures, whose properties become more vivid when they are studied as a group rather than in isolation.

Moreover, Mr. A. R. Pargeter's method of colouring and folding H, described in Note 2449, results in flaps appearing on the outside which have to be turned over in what seems to me to be a most inelegant manner, and this obscures rather than emphasizes (as claimed) the connexion between the two figures. Similar colour changes can be induced in a much smoother and more impressive way by colouring and folding the figure in the original way described by Miss Margaret Joseph in her article in *The Mathematics Teacher* for April 1951. For the benefit of readers who do not have access to this journal, the essential details will be repeated presently.

The word flexagon will in future be used to describe *any* figure which is formed from a plane strip of equilateral triangles by folding them into a hexagon and joining the ends, and which can then be made to exhibit different visible colour or number combinations according to the method of folding. We shall, however, regard H and other flexagons with loose flaps as improper specimens, and exclude them from further consideration. For reasons which will be explained presently, the flexagon F will be called F_1 and Miss Joseph's original "hexahexaflexagram" will be called F_4. The reader may have difficulty in following some of the descriptions below unless he has actually experimented with one or two flexagons. It is recommended, therefore, that the reader make models of F_1 and F_4; he will then understand the behaviour of this type of figure sufficiently to appreciate the transformations of the others without actually constructing *all* the examples mentioned.

F_1 is constructed from a strip S_1 of ten equilateral triangles (one of which is used for joining) and these are coloured as shown with the three primary colours, red, yellow, and blue. The roman letters denote the colours on the front, the italic those on the back, and the Δs represent the two blank faces which are ultimately glued together. S_1 is folded (by giving it three half-twists) so that all the yellow faces, say, are hidden and the blank faces are then

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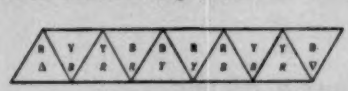
F_1 is constructed from a strip S_1 of ten equilateral triangles (one of which is used for joining) and these are coloured as shown with the three primary colours, red, yellow, and blue. The roman letters denote the colours on the front, the italic those on the back, and the Δs represent the two blank faces which are ultimately glued together. S_1 is folded (by giving it three half-twists) so that all the yellow faces, say, are hidden and the blank faces are then

1

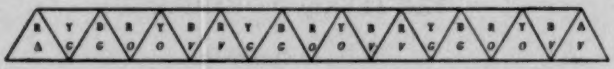
joined. The resulting flexagon F_1 can be opened with any two colours exposed and the third hidden; i.e. the colour changes can be represented by the "map" M_1, a concept which is trivial here, but which will prove most valuable for later developments.



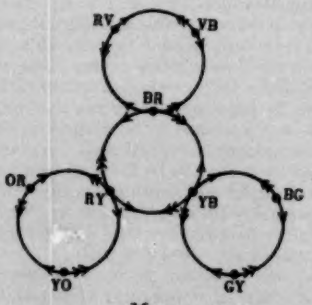
S_1

M_1

F_4 is constructed from a strip S_4 of nineteen equilateral triangles (one of which is redundant). These are coloured as shown, using the three primary colours, plus the three secondary colours, orange, green, and violet. S_4 is first folded by bringing together each pair of adjacent faces which show the same secondary colour. By giving the strip nine half-twists, *all* the secondary colours can be hidden. The resulting strip then looks exactly like S_1 above, and the folding is completed in the same way as for F_1 by giving it three more



S_4



M_4

half-twists. (Contrast H, which had only six half-twists altogether instead of twelve; in fact, any flexagon which has $3t$ triangles and less than $(3t - 6)$ half-twists will be improper.)

It will be found that the possible colour combinations are represented by the map M_4. There is a primary cycle which will display any pair of primary colours, R, Y, B. Branching off from this are three secondary cycles, each of which contains two primary colours and their associated secondary (such as Y, G, B), and any pair of these can be exhibited. It is not possible to expose simultaneously two secondary colours or two complementary colours (such as yellow and violet). If one continues manipulating the flexagon in the same general direction without turning it over, one will follow continuously either the single or the double arrows round the map. When folding the flexagon inwards one can bring together (see Fig. 1) either the three radii a, or the three lines b. In cases where the map branches, it will be found that the flexagon will spring open again from *either* configuration, and this is how one switches from one cycle to another. To pass from, say, violet and blue to orange and

red, one either goes via VB, RV, BR, YB, RY, OR or via VB, BR, RY, YO, OR. [It is also possible to juggle this figure into a triangular dipyramid whose six faces are all coloured alike, and any one of the six colours can be presented in this way.]

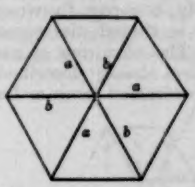
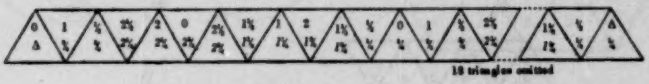
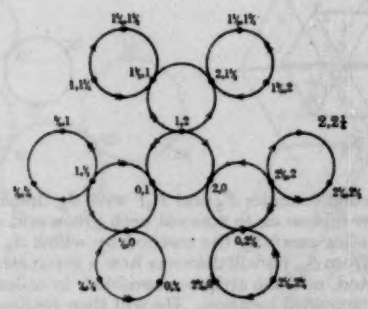


FIG. 1

We now come to our first extension, the flexagon F_{10}. It is formed from the strip S_{10} which has thirty-seven triangles, eighteen of which have been omitted in the diagram to save space. This strip can be coloured by adding to our palette the six tertiary colours, red-orange, orange-yellow, etc. An alternative method, however, which generalizes more neatly is to number the triangles



S_{10}



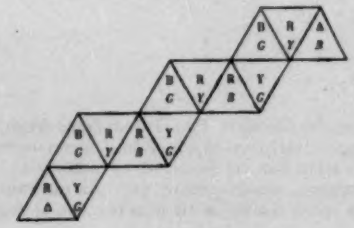
M_{10}

with fractions (mod. 3). The primary cycle uses only the integers 0, 1, and 2, the secondary cycles introduce the halves $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, by associating with each half, h, the integers $h \pm \frac{1}{2}$ (mod. 3), and the tertiary cycles annex the quarters $\frac{1}{4}$, ..., $2\frac{3}{4}$ in an analogous way. Altogether, the map M_{10} contains ten cycles: this is the significance of the suffix in the F_{10} symbol. (As far as I am aware, the *systematic* schemes described here for colouring or numbering the strips are both original.) S_{10} is first folded by concealing all the quarters, and the strip then resembles S_4. The halves are then hidden, giving a strip like S_1, and this is folded and joined in the now familiar way. Warning: if you make F_{10}, use thin cardboard, or your model will be a *fix*agon!

The reader will have no difficulty in seeing how this sequence of flexagons

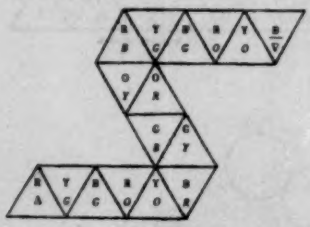
F_1, F_4, F_{10} can be extended,* and in designing the necessary strips. The general member is F_n where $n = 3 \cdot 2^s - 2$.

This sequence contains the most interesting flexagons, for they have a completeness and symmetry which most representatives lack. It does not by any means exhaust the family, however, for we can design at least one proper flexagon F_n for any chosen n, though the necessary strips S_n are no longer straight rows of triangles. The colouring or numbering cannot be done as systematically as for the figures already described; in the remaining examples in this article, it has been performed more or less arbitrarily. The diagrams

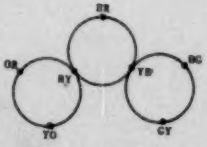


S_2

M_2



S_3



M_3

show the strips needed to make F_2 and F_3; with S_2, first fold so as to hide the green, with S_3, first fold so as to conceal both green and orange.

If the reader studies carefully the method by which S_2 is obtained from S_1, S_3 from S_2, and S_4 from S_3, he will discover how a given strip can be broken up, a new colour inserted, and the strip reassembled, in order to add a new cycle to the map at any specified location. He will then realize that a flexagon can be tailor-made to have *any* given (permissible) map, and the question naturally arises how many different F_n exist for a given n. F_0 is just an ordinary plane hexagon, without true flexagon properties. F_1, F_2, and F_3 are all essentially unique, since the maps M_1, M_2, M_3 are the only ones obtainable using one, two, three cycles respectively. (It is obviously impossible to have a map consisting of three cycles, each pair of which are mutually tangent—why?)

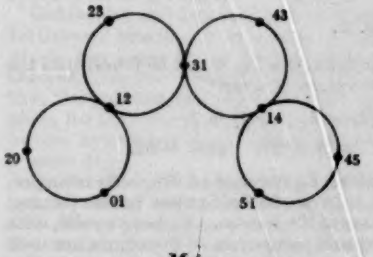
In addition to the map M_4, however, there are two others (M_4' and M_4'', say) which also contain four cycles. These are shown in the diagrams, which also show S_4' and S_4''. Since there is no room left for an extra triangle for joining these strips, the flexagons F_4' and F_4'' must either be joined with

* The existence of these flexagons F_{10}, F_{22}, ... was discovered independently by Edwin F. Ford (now at Harvard University), who was a High School student of mine while I was on exchange in the U.S.A.

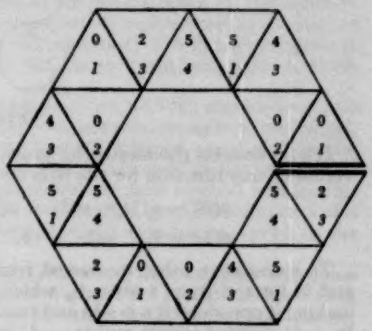
adhesive tape, or built from two halves; the thick lines show the edges which are eventually joined, after giving the necessary twelve half-twists.

To discover the number of distinct maps M_n for a given n, it is necessary to ascertain first if the cycles belonging to a map can ever form a closed ring.

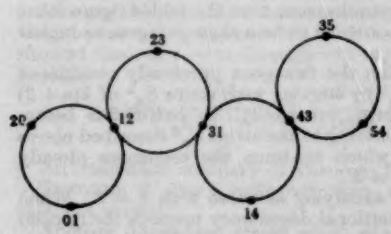
A point on the map which belongs to only one cycle will be called an *isolated point*. When the flexagon is folded flat, the six component triangles of the hexagon fall into two triads, each consisting of alternate triangles of the



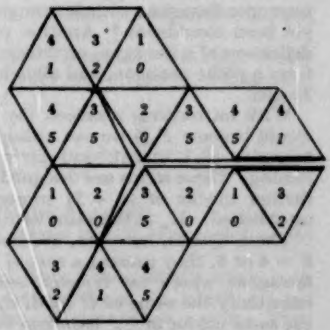
M_4'



S_4'



M_4''



S_4''

hexagon. One of these triads may be of single thickness only, in which case, we shall call it a *layer triad*. The following results are true for any proper flexagon:

1. The flexagon can be manipulated so that any specified triangle belongs to a layer triad.

2. The hexagon will have a layer triad if and only if the colour combination then exposed occurs at an isolated point of the flexagon's map.

3. Hence, there will be triangles of S_n whose two faces are coloured A and B if and only if there is an isolated point on M_n marked AB.

4. This observation enables us to prove that there can be no map in which a succession of cycles forms a closed ring. It is obvious that an *odd* number of cycles cannot form a ring, since the colours cannot be assigned to such a map

consistently. If an *even* number of cycles could form a ring it would have to be coloured or numbered as shown. (The situation is illustrated in Fig. 2 for a four-cycle ring, but is similar for any larger even number.) Since the number 0 occurs at no isolated point, there is no means of marking a triangle with this number, and this establishes the result.

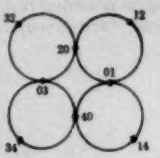


FIG. 2.

If $\varphi(n)$ denotes the number of proper flexagons F_n, it will be found that the values of this function for the first few values of n are*

$$\varphi(0) = \varphi(1) = \varphi(2) = \varphi(3) = 1, \quad \varphi(4) = 3,$$
$$\varphi(5) = 4, \quad \varphi(6) = 12, \quad \varphi(7) = 27, \quad \varphi(8) = 82.$$

To summarize other numerical results: F_n consists of $3(n + 2)$ triangles, and is formed from a strip S_n which is given $3n$ half-twists before joining, making it one-sided if n is odd and two-sided if n is even. M_n has n cycles, with $(n + 2)$ isolated points and $(n - 1)$ branch points; $(n + 2)$ colours are used and $(2n + 1)$ different colour combinations are obtainable.

These studies suggest various other investigations: for instance, the improper flexagons with $3t$ triangles and less than $(3t - 6)$ half-twists have not yet been considered.†· Another possible generalization arises by relaxing our definition of a flexagon, omitting the requirement that the folded figure must form a plane hexagon, and allowing it instead to be a skew octagon, or higher $2k$-gon.

With an obvious notation (by which the flexagons previously considered would become F_n^3) we can obtain F_n^k by starting with strips S_n^k of $k(n + 2)$ triangles ($+1$ extra), and giving them, eventually, kn half-twists before joining. These strips are designed by adding to the strips S_n^3 described above further blocks of $(n + 2)$ triangles which continue the sequences already established. M_n is the same as for $k = 3$.

These figures, however, are not as satisfying as those with $k = 3$. When $k = 4$ or 5, they present a sort of transitional degeneracy towards the flapped flexagons which we rejected earlier. Indeed, F_1^6 (with three colours) is essentially the same as H (with six), for each set of twelve faces coloured with the same colour in F_1^6 falls into two distinguishable groups of six, and if these groups are coloured differently, we recover H exactly. A similar recolouring is possible whenever $k = 3m$, and gives a flapped flexagon with $m(n + 2)$ colours, each occurring six times only.

Hymers College, Hull. ROGER F. WHEELER

* The evaluation of $\varphi(n)$ is the subject of a forthcoming note.

† In Note 2672 (Fb. 1957, p. 55), Miss Joan Crampin has described some of these. Her sequence is of the type referred to in the last paragraph as F_n^{3m}, with $n = 1$. Her note will explain clearly how the recolouring mentioned there can be accomplished when $n = 1$.

ADDITIVE PRIME NUMBER THEORY*

By L. Mirsky

The scope of the additive prime number theory is evident from its name. In this part of the theory of numbers we are concerned with the representation of integers as sums of primes; and here the central problem consists in the proof (or possibly refutation) of a celebrated conjecture made by Goldbach in 1742 to the effect that every even integer >4 can be represented as the sum of 2 primes. The truth of this conjecture is still unsettled; but though progress in this field has been slow, attempts to deal with the problem have led to a whole series of striking results.

Goldbach's problem remained virtually untouched till 1923 when Hardy and Littlewood attacked it by means of a new and powerful analytical technique, which has come to be known (from a contour integral employed in the argument) as the "circle method." The work of Hardy and Littlewood showed that the problem was tractable; but the success of their method was incomplete, for they were only able to establish their results on the assumption of a certain hypothesis in the theory of functions which remains unproved to the present day.†

By combining the ideas of Hardy and Littlewood with daringly ingenious inequalities for exponential sums, Vinogradoff succeeded in 1937 in dispensing with H and putting on a firm foundation all earlier hypothetical statements. The most important result proved in this way was as follows.

THEOREM 1. *Every sufficiently large odd integer can be represented as the sum of 3 primes.*

This result follows, of course, trivially from Goldbach's conjecture, but the converse inference cannot be made.

Vinogradoff proved, in fact, much more than Theorem 1. Let $r_3(n)$ denote the number of representations of n as the sum of 3 primes. Vinogradoff showed that, as $n \to \infty$ through odd values,‡

$$r_3(n) \sim \frac{n^2}{2\log^3 n} \prod_p \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|n} \left(1 - \frac{1}{p^2 - 3p + 3}\right). \quad \ldots\ldots(1)$$

An immediate corollary of Theorem 1 may also be noted.

THEOREM 2. *Every sufficiently large even integer can be represented as the sum of 4 primes.*

Another interesting result which emerged from Vinogradoff's work is as follows. Let $N(x)$ denote the number of even integers $\leq x$ which are *not* representable as the sum of 2 primes; then $N(x)/x \to 0$ as $x \to \infty$. Thus Goldbach's conjecture is true at any rate in the majority of cases, and this assertion is usually expressed in the following form.

THEOREM 3. *Almost all even integers can be represented as sums of 2 primes.*

All three theorems stated above, as well as the asymptotic formula (1), were first established by Hardy and Littlewood on the basis of the unproved hypothesis H.

* Shortened version of an address given to the British Mathematical Colloquium at St. Andrews in September, 1956.

† This hypothesis, which we shall refer to as H, is a generalization of the famous "Riemann hypothesis."

‡ The symbol $\sim$ in (1) indicates that the ratio of the two sides tends to unity. The letter p is reserved for primes throughout; and in the first product on the right-hand side of (1) p ranges over all primes, while in the second it ranges over all prime divisors of n.

The development of the circle method at the hands of Hardy, Littlewood, and Vinogradoff is probably comparatively well known, and we therefore propose to consider in greater detail an alternative treatment of problems in the additive theory of numbers. This is the "sieve method," which was devised originally by Viggo Brun in 1915–20 and has since been modified and extended by other writers. The sieve method is elementary and does not lead to the complete solution of any problem but, compared with the circle method, it has the advantage of being applicable to a wider range of questions.

Consider the sequence of integers 1, 2, ..., n and let $p_1, \ldots, p_r$ be any r distinct primes. From this sequence we delete all integers which belong to at least one of r specified arithmetic progressions with common differences $p_1, \ldots, p_r$ respectively, i.e. we delete one residue class (mod p_1), one residue class (mod p_2), and so on. More generally, it may be necessary to delete $m > 1$ residue classes modulo each of the primes $p_1, \ldots, p_r$, and this procedure is described as "sieving." Brun gave a method for estimating the number of integers in the sequence 1, 2, ..., n which survive the process of sieving, and he showed how such estimates could be used for dealing with questions which had previously seemed intractable.

An adaptation of Brun's sieve method enabled Schnirelmann, in 1930, to make a very substantial advance in the direction of Goldbach's conjecture. Denote by $p_1, \ldots, p_k$ all the primes $\leqslant \sqrt{n}$, and by $Q(n)$ the number of positive integers $t < n$ such that

$$t \not\equiv 0, \qquad t \not\equiv n \pmod{p_i} \qquad (i = 1, \ldots, k).$$

Thus $Q(n)$ is the number of integers $<n$ which do not belong to two specified residue classes modulo each of the primes $p_1, \ldots, p_k$. Using the sieve method, Schnirelmann showed that*

$$Q(n) < c_1 \frac{n}{\log^2 n} \sum_{d|n} d^{-1}, \qquad \ldots\ldots(2)$$

where in the summation d ranges over all divisors of n. Next, let $r_2(n)$ denote the number of representations of n as the sum of two primes, i.e. the number of solutions of the equation

$$n = p' + p''. \qquad \ldots\ldots(3)$$

The number of solutions of (3), subject to $p' > \sqrt{n}$, $p'' > \sqrt{n}$, does not exceed $Q(n)$; while the number of solutions subject to $\min(p', p'') \leqslant \sqrt{n}$ does not exceed $2\sqrt{n}$. Hence

$$r_2(n) < Q(n) + 2\sqrt{n},$$

and so, by (2),

$$r_2(n) < c_2 \frac{n}{\log^2 n} \sum_{d|n} d^{-1}.$$

Thus $r_2(n)$ is never very large. On the other hand, the average value of $r_2(n)$ is not too small, for

$$\sum_{n \leqslant x} r_2(n) = \sum_{p'+p'' \leqslant x} 1 > c_3 \frac{x^2}{\log^2 x}.$$

Now if a non-negative function is not too small on the average and not too large for any value of the variable n, then it must be positive for many values of n. The precise formulation of this idea enables us to infer that the number $E(n)$ of integers $t < n$ such that $r_2(t) > 0$ (i.e. the number of integers $<n$ representable in the sense of Goldbach) satisfies the inequality $E(n) > c_4 n$.

* The letter c is reserved for absolute positive constants.

In other words, the sequence of "representable" integers has positive density.* Now it was shown by Schnirelmann—and the proof is not at all difficult—that if a sequence $\{s_i\}$ of integers has positive density δ, then there exists a number $l = l(\delta)$ such that every integer can be represented as the sum of at most l elements of $\{s_i\}$. It follows, therefore, that every integer is the sum of at most c_8 "representable" integers, and so the sum of at most $2c_8 = c_9$ primes.

The value of c_9 which emerged from Schnirelmann's analysis was large, namely 800,000. Nevertheless, the theorem that every integer is the sum of at most 800,000 primes constituted a very large step forward, since at the time only the hypothetical results of Hardy and Littlewood were available. Subsequent writers improved on the quantitative aspect of Schnirelmann's work. It was shown, eventually, that every sufficiently large number is the sum of at most 67 primes, but even this result was soon superseded by Vinogradoff's Theorems I and II.

So far we have been discussing the representation of integers as sums of more than 2 primes, but a relaxation of Goldbach's conjecture can clearly take different forms and we might, for example, consider the representation of n in the form $n = n_1 + n_2$, where the conditions imposed on n_1 and n_2 are less severe than those of primality. Brun employed his sieve method in an investigation of this type and showed that every sufficiently large integer can be represented as the sum of 2 integers each of which has at most 9 prime divisors. This result was sharpened in successive stages till, in 1940, Buchstab proved

THEOREM 4. *Every sufficiently large integer can be represented as the sum of* 2 *integers each of which has at most* 4 *prime divisors.*†

A variant of the problem just considered consists in investigating the existence of representations of n in the form $n = p + n'$, where n' has few prime divisors. Already in 1932 Estermann had shown that, subject to a hypothesis slightly stronger than H, every sufficiently large integer n possesses such a representation, with n' having at most 6 prime divisors. In 1948 Renyi proved an extraordinarily interesting theorem which, though weaker than Estermann's, does not depend on any unproved result. Renyi's theorem is as follows.

THEOREM 5. *Every integer can be represented as the sum of a prime and a number having at most* c_7 *prime divisors.*

The proof of this result is long and difficult and makes full use both of arithmetical and analytical techniques. Probably the most interesting feature of the proof is a powerful generalization of Brun's sieve method, and we shall conclude by briefly referring to this aspect of Renyi's work. It will be recalled that the sieve method, as devised by Brun, is concerned with estimating the number h of integers $\leqslant n$ which do not belong to m specified residue classes modulo each of the primes $p_1, \ldots, p_r$. Consider now the more general case when the number of excluded residue classes (mod p_i), instead of being a fixed number m, is some function $f(p_i)$ of p_i. Brun's method for estimating h is now no longer applicable, but in 1941 Linnik invented a technique for dealing with this new operation—an operation which he aptly termed 'the large sieve.' Linnik's theorem states that *if* $p_1, \ldots, p_r$ *are any primes* $< \sqrt{n}$ *and* $\sigma = \min f(p_i)/p_i$, *then* $h < c_8 n/\sigma^2 r$. Even this result was insufficient for Renyi's purpose and a yet further generalization of the sieve method was needed before Theorem 5 could be established.

There are, of course, results bearing on Goldbach's conjecture other than those we have mentioned. Thus, for example, a rather amusing theorem due to

* The density of the sequence $\{s_i\}$ is defined as the lower bound of $S(n)/n$, where $S(n)$ denotes the number of s_i which do not exceed n.

† A. Selberg stated in 1949 that it is possible to replace the number 4 in Theorem 4 by 3, but he has not published a detailed proof.

Linnik (1953) states that if any even integer is written in the binary scale, then it is possible to convert it into a "Goldbach number" by changing at most c_9 digits. However, the five theorems listed above are certainly the most important known approximations to Goldbach's conjecture. In a sense they all come near to the desired end, yet the final proof still eludes us; and it would probably be rash to hazard a guess whether we are as yet anywhere near the definitive solution of the problem.

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1900. But, my dear Mrs. Witham, indeed you need not be concerned about me! A man who is reading for the Mathematical Tripos has too much to think of to be disturbed by any of these mysterious "somethings," and his work is of too exact and prosaic a kind to allow of his having any corner in his mind for mysteries of any kind. Harmonical Progressions, Permutations and Combinations, and Elliptic Functions have sufficient mysteries for me!—Bram Stoker, *The Judge's House*. [Per Mr. B. Cook.]

1901. It is laid down that the smallest offence against God being committed in the sight of an infinitely wise and good being, "who is of purer eyes than to behold iniquity," must appear like a spot of the deepest dye, must cause an infinite degree of disapprobation, and subject the offender to an unmitigated and never-ending punishment in consequence.... Can anyone suppose that Sir Isaac Newton, with the accessions of knowledge which his mild spirit may have acquired in the starry sphere, would be thrown into convulsions of rage and agony on beholding some mistake which a poor village schoolmaster had committed in a sum in arithmetic.—William Hazlitt, *Complete Works* (Centenary Edition) XX, p. 225. [Per Professor T. A. A. Broadbent.]

1902. In order to prove that a guinea and a feather would descend *in vacuo* in the same time, he (Isaac Milner, Lucasian Professor at Cambridge, 1798–1820) made use of a glass tube hermetically sealed in, in which the guinea and the feather were enclosed; it so happened, that in several attempts the guinea had the advantage; he then managed to place the guinea above the feather. At the end he exclaimed, "How beautifully this experiment has succeeded! For if you observed attentively, you would perceive that the feather was down sooner than the guinea".—H. Gunning, *Reminiscences of the University, Town and Country of Cambridge*, (2nd edition, 1855), Vol. I, p. 236. [Per Professor T. A. A. Broadbent.]

1903. Accounts under the Metric System. Accordingly the gross was adopted as the basic unit with prices quoted in shillings and decimals of a shilling. A single article becomes 0·007 of a gross, a penny is reckoned as 0·083 of a shilling, and only the total price of a quotation is given in £ s. d. Thus the customer now receives an invoice which instead of quoting say three gross, seven dozen articles at £2 10s. 6d. a gross, refers to 3·583 gross at 50·5 shillings a gross

So far only once customer has complained. On receiving his first invoice he telephoned to say he could not understand it. For his special benefit quantities and prices in the old reckoning are now inserted parenthetically on invoices sent to his firm.—*The Times*, 14th March 1957. [Per Mr. W. H. Cozens.]

1904. Question and Answer. Can a sailing boat sail faster than the wind?—*Arthur Pells, Cowbridge road, Cardiff.*

Not if the wind is astern, but it can sail much faster than the wind if this is at right angles to the boat's course.—*Daily Herald*, 25th April 1957. [Per Mr. W. H. Cozens.]

THE MOMENTS OF INERTIA OF SOME POLYHEDRA

By John Satterly

The values of the Radii of Gyration have been worked out for some solid polyhedra, polyhedra with faces only, (shell or box-like), polyhedra with edges only (skeleton), polyhedra with vertices only. The results are given in the tables.

The usual definition of the radius of gyration, k, of a body about any given axis is $k^2 = (\Sigma mr^2)/(\Sigma m) = I/M$ where m is an element of mass of the body, r the distance of this element from the given axis. M is the total mass $= \Sigma m$ and I is the moment of inertia of the body about the given axis. In the cases of polyhedra with either faces only, edges only or vertices only, there is theoretically no mass but we can say for these cases that $k^2 =$ Limit, as m approaches zero, of $(\Sigma mr^2)/(\Sigma m)$.

In the tables a is the length of the edge of the polyhedron and R_v, R_e, R_f are the radii of the concentric spherical surfaces which respectively pass through the vertices, the edges and touch the faces.

All the results have been worked out *de novo* from the external appearance of the polyhedra and possibly some of them are new. The geometry was first established. The solid polyhedra were supposed divided up into groups of pyramids having as bases the faces of the polyhedra and a common vertex at the centre of the polyhedron. The known rules for the calculation of moments of inertia were then applied. In the other cases the faces, edges or vertices were treated as plane areas, straight lines or points. The distribution of mass was taken as uniform throughout the solid, or over the faces, or along the edges and among the vertices respectively.

Table 1 gives, for the polyhedra studied, the values of the three radii and

Table 1

Polyhedron	R_v^2/a^2	R_e^2/a^2	R_f^2/a^2	V/a^3
Tetrahedron	3/8	1/8	1/24	$\sqrt{2}/12$
Octahedron	1/2	1/4	1/6	$\sqrt{2}/3$
Cube	3/4	1/2	1/4	1
Icosahedron	$\tau\sqrt{5}/4$	$\tau^2/4$	$\tau^4/12$	$\tau^2 5/6$
Pentagonal-dodecahedron	$\tau^2 3/4$	$\tau^4/4$	$\tau^5/4\sqrt{5}$	$\tau^4\sqrt{5}/2$
Rhombic-dodecahedron	$\begin{cases}(a)\ 4/3\\(o)\ 1\end{cases}$	8/9	2/3	$16\sqrt{3}/9$
Truncated-octahedron	5/2	9/4	$\begin{cases}(s)\ 2\\(h)\ 3/2\end{cases}$	$8\sqrt{2}$
Rhombicub-octahedron	$5/4 + 1/\sqrt{2}$	$1 + 1/\sqrt{2}$	$\begin{cases}(s)\ 3/4+1/\sqrt{2}\\(t)\ 11/12+1/\sqrt{2}\end{cases}$	$4 + 10\sqrt{2}/3$

Notes.

(a) = to acute vertex, (o) = to obtuse vertex, (s) = to square face, (h) = to hexagonal face, (t) = to triangular face.

$\tau = (\sqrt{5} + 1)/2 = 2\cos 36° =$ positive root of equation $x^2 - x - 1 = 0 = 1{\cdot}615034$. It has interesting numerical properties such as $\tau^2 = \tau + 1$. $1/\tau = \tau - 1$ which may be written and read as τ to the power minus one $= \tau$ minus one, etc.

Also $\tau^3 = \tau + 1 = 2{\cdot}618034\ldots$, $\tau^3 = 2\tau + 1 = 4{\cdot}236068\ldots$,

$\tau^4 = 3\tau + 2 = 6{\cdot}854102\ldots$, $\tau^5 = 5\tau + 3 = 11{\cdot}090170\ldots$

the volumes in terms of the edges. For polyhedra which have the pentagon as an intrinsic part of the structure the results are expressed in terms of τ. The values of τ and of some important expressions containing τ are given underneath Table 1.

Table 2 gives, in the upper half, the values of k^2 in terms of a^2 and in the lower half the values of k^2 in terms of R_e^2. In most cases the expressions are given in exact algebraical form and in addition in some cases in decimal form to at least four places of decimals.

For any given polyhedron the value of k^2 is the same about all axes passing through the centre.

TABLE 2

Values of k^2/a^2

The regular polyhedron	Solid	Faces only (i.e. shell)	Edges only (i.e. skeleton)	Vertices only
Tetrahedron	1/20	1/12	5/36	1/4
Octahedron	1/10	1/6	2/9	1/3
Cube	1/6	5/18	7/18	1/2
Icosahedron	$\tau^2/10 = 0{\cdot}2618$	$\tau^2/6 = 0{\cdot}4363$	$(3\tau+4)/18$ $= 0{\cdot}4918$	$(\tau+2)/6$ $= 0{\cdot}6030$
Pentagonal-dodecahedron	$(78\tau+56)/300$ $= 0{\cdot}6074$	$(39\tau+28)/90 = 1{\cdot}0123$	$(52\tau+14)/90$ $= 1{\cdot}0904$	$\tau^2/2 = 1{\cdot}3090$
Rhombic-dodecahedron	$19/60 = 0{\cdot}3167$	$5/9 = 0{\cdot}5556$	$2/3 = 0{\cdot}6667$	$16/21 = 0{\cdot}7619$
Truncated octahedron	$19/24 = 0{\cdot}7917$	$(125+3\sqrt{3})/99 = 1{\cdot}3151$	$14/9 = 1{\cdot}5556$	$5/3 = 1{\cdot}6667$
Rhombicub-octahedron	$(100+59\sqrt{2})/280$ $= 0{\cdot}65514$	$(95+52\sqrt{2}+\sqrt{3})/156$ $= 1{\cdot}09148$	$(13+6\sqrt{2})/18$ $= 1{\cdot}1936$	$(5+2\sqrt{2})/6$ $= 1{\cdot}3047$

Values of k^2/R_e^2

Tetrahedron	2/5	2/3	10/9	2
Octahedron	2/5	2/3	8/9	4/3
Cube	1/3	5/9	7/9	1
Icosahedron	2/5	2/3	$2/9(5-\tau)$	$2/3(3-\tau)$
Pentagonal-dodecahedron	$(46-12\tau)/75$ $= 0{\cdot}3544$	$(92-24\tau)/90 = 0{\cdot}5908$	$8(31\tau-43)/90$ $= 0{\cdot}6364$	$2/\tau^2 = 0{\cdot}7639$
Rhombic-dodecahedron	$57/160 = 0{\cdot}3562$	$5/8 = 0{\cdot}6250$	$3/4 = 0{\cdot}7500$	$6/7 = 0{\cdot}8571$
Truncated-octahedron	$19/54 = 0{\cdot}3519$	$(500+12\sqrt{3})/891 = 0{\cdot}5845$	$56/81 = 0{\cdot}6914$	$20/27 = 0{\cdot}7407$
Rhombicub-octahedron	$(82+18\sqrt{2})/280$ $= 0{\cdot}3838$	$(86+9\sqrt{2}+2\sqrt{3}-\sqrt{6})/156$ $= 0{\cdot}6394$	$(14-\sqrt{2})/18$ $= 0{\cdot}6992$	$(6-\sqrt{2})/6$ $= 0{\cdot}7643$

Observations: (1) The value of k^2 increases as we pass from solid, to faces only, to edges only, to the vertices only. This is fairly obvious. In the case of the cube the four values of k^2 are in Arithmetical Progression.

(2) k^2 about a diameter for a solid sphere $= \frac{2}{5}R^2$ and for a spherical shell $k^2 = \frac{1}{3}R^2$. The ratio k^2 solid/k^2 shell $= 3/5$. For the tetrahedron, octahedron, icosahedron—the three polyhedra having equilateral triangles as their faces—k^2 solid $= \frac{2}{5}R_e^2$ and k^2 shell $= \frac{1}{3}R_e^2$ and therefore k^2 solid/k^2 shell $= 3/5$ also.

For the cube k^2 solid/k^2 shell $= 3/5$ also.

For the truncated octahedron and the rhombicub-octahedron the expressions for k^2 are complicated but k^2 solid/k^2 shell is very nearly equal to 3/5, being 0·60197 for the truncated octahedron and 0·60023 for the rhombicuboctahedron.

The rhombic dodecahedron is unique. For it the ratio k^2 solid/k^2 shell $= 0{\cdot}57$, exactly.

The values of k^2 for all these solid polyhedra and for some of the shell and skeleton polyhedra have been confirmed with experiments on models placed on the inertia frame of a torsion pendulum. The model of the octahedron was a

large alum crystal, the others were made of wood, glass or brass. The agreement between theory and experiment is as good as could be expected.

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A FACTORIAL SERIES FOR THE RATIONAL MULTIPLES OF e

By K. Mahler

A special case of a theorem by G. Cantor* states that every real number α can be written in a unique way as a series

$$\alpha = \sum_{n=1}^{\infty} \frac{g_n}{n!} \qquad \text{.....(1)}$$

where the coefficients g_n are integers, g_1 being arbitrary, while

$$0 \leqq g_n \leqq n-1 \quad \text{for all } n > 2 \qquad \text{.....(2)}$$

and

$$0 \leqq g_n \leqq n-2 \quad \text{for infinitely many } n > 2. \qquad \text{.....(3)}$$

One finds, in fact, that

$$g_1 = [\alpha], \quad \text{and} \quad g_n = [n!\,\alpha] - n[(n-1)!\,\alpha] \quad \text{for } n > 2,$$

and that, more precisely,

$$\alpha = \sum_{n=1}^{N} \frac{g_n}{n!} + \frac{\alpha_N}{N!}$$

where

$$\alpha_N = N!\,\alpha - [N!\,\alpha] = N! \sum_{n=N+1}^{\infty} \frac{g_n}{n!}, \quad 0 \leqq \alpha_N < 1.$$

Our aim is to construct the series (1) in the special case when α is a rational multiple of e. For simplicity we shall, however, assume that

$$\alpha = \frac{p}{q} e, \text{ where } p \text{ and } q \text{ are integers, and } 1 < p < q - 1. \qquad \text{.....(4)}$$

The developments of other rational multiples of e may be obtained by adding suitable integral multiples of one of the series

$$e = 2 + \sum_{n=2}^{\infty} \frac{1}{n!}, \quad -e = -3 + \sum_{n=3}^{\infty} \frac{n-2}{n!}.$$

1. The classical series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

may be written as

$$e = \sum_{m=0}^{\infty} \sum_{k=0}^{q-1} \frac{1}{(mq+k)!}.$$

We therefore shall try to find integers a_k, b_k such that

$$\frac{p}{q} \sum_{k=0}^{q-1} \frac{1}{(mq+k)!} = \sum_{k=0}^{q-1} \frac{a_k m + b_k}{(mq+k+1)!} \qquad \text{.....(5)}$$

identically in m. For this identity implies that

$$\frac{p}{q}e = \sum_{m=0}^{\infty}\sum_{k=0}^{q-1}\frac{a_k m + b_k}{(mq+k+1)!}, \qquad \text{.....(5a)}$$

giving the required series, provided that

$$0 \leqslant a_k m + b_k < mq + k \qquad \text{.....(6)}$$

for all pairs of integers k, m with $mq + k > 1$, and

$$0 < a_k m + b_k < mq + k - 1 \qquad \text{.....(7)}$$

for infinitely many such pairs.

2. The identity (5) is equivalent to

$$\sum_{k=0}^{q-1}\frac{p}{(mq+k)!} = \sum_{k=0}^{q-1}\frac{(a_k m + b_k)q}{(mq+k+1)!} = \sum_{k=0}^{q-1}\left\{\frac{a_k}{(mq+k)!} + \frac{b_k q - (k+1)a_k}{(mq+k+1)!}\right\}$$

$$= \frac{a_0}{(mq)!} + \sum_{k=1}^{q-1}\frac{a_k + b_{k-1}q - ka_{k-1}}{(mq+k)!} + \frac{(b_{q-1} - a_{q-1})q}{(mq+q)!}.$$

It is therefore satisfied if

$$a_0 = p,$$
$$a_k + b_{k-1}q - ka_{k-1} = p \qquad (k = 1, 2, \ldots, q-1),$$
$$b_{q-1} = a_{q-1}.$$

It thus suffices to choose

$$a_k = \begin{cases} p & \text{if } k = 0, \\ p + ka_{k-1} - b_{k-1}q = (p + ka_{k-1}) - \left[\dfrac{p + ka_{k-1}}{q}\right]q & \text{if } k = 1, 2, \ldots, q-1 \end{cases} \qquad \text{.....(8)}$$

and

$$b_k = \begin{cases} \left[\dfrac{p + (k+1)a_k}{q}\right] & \text{if } k = 0, 1, \ldots, q-2, \\ a_{q-1} & \text{if } k = q-1. \end{cases} \qquad \text{.....(9)}$$

3. Since $1 < p < q - 1$, evidently

$$0 < a_k < q - 1 \qquad (k = 0, 1, \ldots, q-1). \qquad \text{.....(10)}$$

Further

$$0 < b_k < k + 1 \qquad (k = 0, 1, \ldots, q-1). \qquad \text{.....(11)}$$

For $b_{q-1} = a_{q-1}$, and so this inequality holds for $k = q - 1$; if, however, $k = 0, 1, \ldots, q-2$, then

$$0 < b_k < \frac{p + (k+1)a_k}{q} < \frac{(q-1) + (k+1)(q-1)}{q} < k + 2,$$

hence $< k + 1$.

From (10) and (11),

$$0 < a_k m + b_k < (q-1)m + (k+1) = (qm + k) - (m - 1).$$

Hence the condition (6) is certainly satisfied when $m > 1$ and the condition (7) when $m > 2$. It follows that all but the q terms

$$\sum_{k=0}^{q-1}\frac{b_k}{(k+1)!} \qquad \text{.....(12)}$$

of the series (A) corresponding to $m = 0$, $k = 0, 1, \ldots, q-1$ have the required form, and this series gives the development (1) for $(p/q)e$ except perhaps for its first q terms. We have thus the following result.

THEOREM 1: *Let $1 < p < q-1$. In the development* (1) *for $(p/q)e$ all but the first q coefficients g_n have the explicit form*

$$g_n = a_k m + b_k \quad \text{if} \quad n = mq + k + 1,\ k = 0, 1, \ldots, q-1,\ m > 1, \quad \ldots\ldots(13)$$

where a_k and b_k are defined by the recursive formulae (8) *and* (9).

In other words, all but finitely many of the coefficients g_n form q separate arithmetic progressions when n runs over the different residue classes (mod q).

4. In addition to the recursive formulae (8) and (9), there are also explicit expressions for a_k and b_k.

Put

$$c_k = k!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!}\right) \quad (k = 0, 1, 2, \ldots), \qquad \ldots\ldots(14)$$

so that c_k is a positive integer, and

$$c_0 = 1, \qquad c_k = 1 + kc_{k-1} \quad \text{if} \quad k > 1. \qquad \ldots\ldots(15)$$

Then, by (8), the expression

$$d_k = a_k - pc_k$$

satisfies the congruence

$$d_k \equiv (p + ka_{k-1}) - p(1 + kc_{k-1}) \equiv kd_{k-1} \pmod{q}.$$

Since evidently $d_0 = 0$, this implies for all $k > 0$ that $d_k \equiv 0 \pmod{q}$ and therefore that

$$a_k \equiv pc_k \pmod{q}.$$

But then, by (10), necessarily

$$a_k = pc_k - \left[\frac{p}{q}c_k\right]q \qquad \ldots\ldots(16)$$

for all values of $k > 0$.

Next, on substituting this expression for a_k in (9), we find that

$$b_k = \left[\frac{p}{q} + \frac{k+1}{q}pc_k - (k+1)\left[\frac{p}{q}c_k\right]\right]$$

and hence that

$$b_k = \left[\frac{p}{q}(1 + (k+1)c_k)\right] - (k+1)\left[\frac{p}{q}c_k\right] \qquad \ldots\ldots(17)$$

for all $k > 0$, including the case when $k = q-1$ because then the right-hand side is equal to

$$\left[\frac{p}{q}\right] + pc_{q-1} - q\left[\frac{p}{q}c_{q-1}\right] = a_{q-1}, \qquad \text{since } \left[\frac{p}{q}\right] = 0.$$

The integers c_k increase rapidly. Therefore it proves to be preferable to use the recursive formulae (8) and (9) rather than the explicit expressions (16) and (17) for the actual computation of a_k and b_k. It may have some interest to study the arithmetical properties of these coefficients.

5. The following two tables give, (i) the lowest cases of the series (A), and (ii) a table of the coefficients a_k and b_k.

Table of series:

$$\frac{e}{2} = \frac{1}{1!} + \sum_{m=1}^{\infty} \frac{m+1}{(2m+1)!},$$

$$\frac{e}{3} = \frac{1}{2!} + \frac{2}{3!} + \sum_{m=1}^{\infty} \left(\frac{m}{(3m+1)!} + \frac{2m+1}{(3m+2)!} + \frac{2m+2}{(3m+3)!} \right),$$

$$\frac{2e}{3} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{2m+1}{(3m+1)!} + \frac{m+1}{(3m+2)!} + \frac{m+1}{(3m+3)!} \right),$$

$$\frac{e}{4} = \frac{1}{2!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{m}{(4m+1)!} + \frac{2m+1}{(4m+2)!} + \frac{m+1}{(4m+3)!} \right),$$

$$\frac{3e}{4} = \frac{2}{1!} + \sum_{m=1}^{\infty} \left(\frac{3m+1}{(4m+1)!} + \frac{2m+1}{(4m+2)!} + \frac{3m+3}{(4m+3)!} \right),$$

$$\frac{e}{5} = \frac{1}{2!} + \frac{1}{4!} + \sum_{m=1}^{\infty} \left(\frac{m}{(5m+1)!} + \frac{2m+1}{(5m+2)!} + \frac{m+1}{(5m+4)!} \right),$$

$$\frac{2e}{5} = \frac{1}{1!} + \frac{2}{4!} + \sum_{m=1}^{\infty} \left(\frac{2m}{(5m+1)!} + \frac{4m+2}{(5m+2)!} + \frac{2m+2}{(5m+4)!} \right),$$

$$\frac{3e}{5} = \frac{1}{1!} + \frac{1}{2!} + \frac{3}{4!} + \sum_{m=1}^{\infty} \left(\frac{3m+1}{(5m+1)!} + \frac{m+1}{(5m+2)!} + \frac{3m+3}{(5m+4)!} \right),$$

$$\frac{4e}{5} = \frac{2}{1!} + \frac{1}{3!} + \sum_{m=1}^{\infty} \left(\frac{4m+1}{(5m+1)!} + \frac{3m+2}{(5m+2)!} + \frac{4m+4}{(5m+4)!} \right).$$

Table of coefficients:

$\frac{p}{q}=$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{7}{10}$	$\frac{9}{10}$
a_0	1	1	2	1	3	1	2	3	4	1	5	1	2	3	4	5	6	1	3	5	7	1	2	4	5	7	8	1	3	7	9
b_0	1	0	1	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	0	1	1	0	0	0	1	1	1	0	0	1	1
a_1	0	2	1	2	2	2	4	1	3	2	4	2	4	6	1	3	5	2	6	2	6	2	4	8	1	5	7	2	6	4	8
b_1	0	1	1	1	1	1	2	1	2	0	2	0	1	2	0	1	2	0	1	1	2	0	1	2	0	1	2	0	1	1	2
a_2		2	1	1	3	0	0	0	0	5	1	5	3	1	6	4	2	5	7	1	3	5	1	2	7	8	4	5	5	5	5
b_2		2	1	1	3	0	0	0	0	2	1	2	1	0	3	2	1	2	3	1	2	1	0	1	2	3	2	1	1	2	2
a_3			0	0	1	2	3	4	4	2	2	4	6	1	3	5	0	0	0	0	7	5	1	8	4	2	6	8	2	4	
b_3			0	0	1	2	3	4	2	2	1	2	3	1	2	3	0	0	0	0	3	2	0	4	2	1	2	3	1	2	
a_4				0	0	0	0	5	1	2	4	6	1	3	5	1	3	5	7	2	4	8	1	5	7	5	5	5	5	5	
b_4				0	0	0	0	4	1	1	3	4	1	2	4	0	2	3	5	1	2	4	1	3	4	2	2	3	3		
a_5							2	4	4	1	5	2	6	3	6	2	6	2	2	4	8	1	5	7	6	8	2	4			
b_5							2	4	3	1	4	2	5	3	4	1	5	2	1	2	5	1	4	5	3	5	1	3			
a_6									4	1	5	2	6	3	5	7	1	3	4	8	7	2	1	5	7	1	9	3			
b_6									4	1	5	2	6	3	4	6	1	3	3	6	5	2	1	4	5	1	7	3			
a_7															4	4	4	4	2	4	8	1	5	7	0	0	0	0			
b_7															4	4	4	4	1	3	7	1	5	7	0	0	0	0			
a_8																			8	7	5	4	2	1	1	3	7	9			
b_8																			8	7	5	4	2	1	1	3	7	9			
a_9																									0	0	0	0			
b_9																									0	0	0	0			

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1905. Few people, I think, realized that (Belloc) was a considerable mathematician, but you were aware of it when you heard him talk about the technical details of bridges or about squaring the circle.—J. B. Morton, *Hilaire Belloc: a memoir*, (Hollis and Carter, 1955), p. 39. [Per Professor T. A. A. Broadbent.]

PEDAL CIRCLES AND THE QUADRANGLE

By H. V. Mallison

In two articles in the *Mathematical Gazette, Pedal Triangles and Pedal Circles*, July 1919, and *Some Properties Relative to a Tetrastigm*, Oct. 1920, J. H. Lawlor gave proofs of a number of properties of the pedal circles of the four vertices of a plane quadrangle $ABCD$ with respect to the triangles formed by the remaining three vertices. The articles concluded with the statement of ten results, the first nine of which were fairly easily deducible from those already proved, but No. (10) was given as a conjecture. This was that the common chord of the pedal circles of A and D bisects BH_1 and CH_2, where H_1 and H_2 are the orthocentres of the triangles BAD and CAD, and hence that BH_2, CH_1 meet on the common chord of the pedal circles of A and D. It can be seen that the second result would imply the former, for if h is the rectangular hyperbola through $ABCD$, which is unique unless the points are orthocentric, and if O is the centre of h, then H_1 and H_2 also lie on h, and it is known that the pedal circles pass through O. The meet of BH_2, CH_1 is a diagonal point of the quadrangle BH_1H_2C inscribed in h, and its join to the centre of h is the diameter conjugate to the parallel chords BH_1, CH_2 and therefore bisects them.

In fact this conjecture is correct, and for an analytical proof we can use a method developed in an article on *Isogonal Conjugates and the Isoptic Point* by P. W. Wood in the *Mathematical Gazette*, Dec. 1941, which also gives other properties of pedal circles.

The equation of the rectangular hyperbola h is taken as $xy = 1$, or $x = t$, $y = 1/t$, and the parameters of A, B, C, D are $t = a, b, c, d$.

It is then proved that the isogonal conjugate of D for the triangle ABC is given by

$$x = \frac{a + b + c - d}{1 - abcd}, \qquad y = \frac{abc - d(bc + ca + ab)}{1 - abcd},$$

The centre of the pedal circle of D is midway between D and its isogonal conjugate, and from the fact that the pedal circle passes through O, its equation is seen to be

$$(1 - abcd)(x^2 + y^2) - (a + b + c - abcd^2)x - \{1/d - d(bc + ca + ab)\}y = 0.$$

Interchanging d and a gives the equation of the pedal circle of A, and the radical axis of the pedal circles of D and A is given by

$$adx + y = 0.$$

This is conjugate to $adx - y = 0$, which is perpendicular to AD, whose equation is $x + ady = a + d$.

Hence the common chord of the pedal circles of A and D bisects the segments BH_1 and CH_2, as was stated.

If the centres of the pedal circles of A, B, C, D are A_1, B_1, C_1, D_1 the above result shows that AD, A_1D_1 are equally inclined to either asymptote of h, and as this is true for any other corresponding pairs of points of the two sets, we get the result that the quadrangles $ABCD$ and $A_1B_1C_1D_1$ are similar.

This interesting property is not given in any of the above articles, and may be compared with the fact proved in the first article that the pedal triangles of A, B, C, D are similar. It may be verified analytically, and it is found that corresponding sides of $A_1B_1C_1D_1$ and $ABCD$ have lengths in the ratio

$$(1 + abcd) : 2(1 - abcd).$$

In J. H. Lawlor's first article it is proved that the radii of the pedal circles

are inversely proportional to the circumradii of the corresponding triangles.

In the search for a synthetic proof of these results a number of interesting properties of the above figure have come to light, chiefly through a correspondence with Mr. T. McHugh, to whom the following proof is due.

Labelling the points as before, let c be the circle with diameter AD, having M the mid-point of AD as centre; and let c meet BA, BD, CA, CD again at

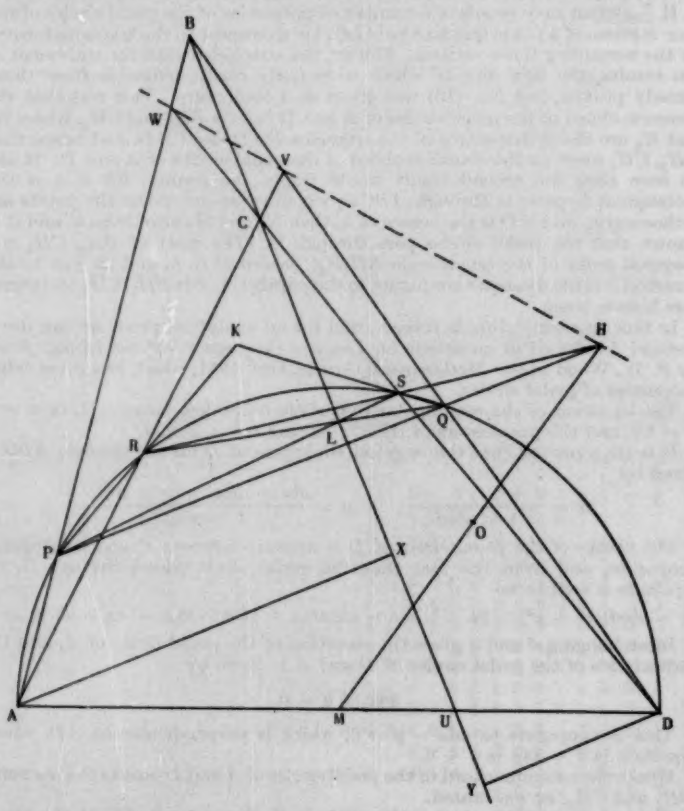

FIG. 1

P, Q, R, S. AX and DY are the perpendiculars from A and D on BC; PQ, RS meet at H; PR, QS meet at K, and PS, QR meet at L.

Then by Pascal's theorem applied to $APSDQR$ we see that L lies on BC.

Also from cyclic quadrangles,

$$\angle RPS = \angle RDS = \angle RYC,$$

since R and Y are on the circle diameter DC; and therefore L lies on the circle PRY, which is the pedal circle of D.

Similarly L lies also on the pedal circle of A.

The point K is the meet of the common chords of these two pedal circles with the circle c, and hence lies on their own common chord.

Thus KL is the radical axis of the pedal circles of A and D—a result given in J. H. Lawlor's first article.

Now let s, t denote the circles BPQ, CRS, their centres being O_1 and O_2. Since K, L are conjugate points for c one of them must lie outside c; let K be this one, and let k be the circle with centre K cutting c orthogonally.

Then by a well-known theorem on orthocentric points the circle c cuts orthogonally both the circles s and t. Hence the tangents from M to the circles s, t, k are all equal in length.

Again H is the radical centre of c, s, t; and as HL is the polar of K for c it is the common chord of the orthogonal circles k, c.

Hence the lengths of tangents (or powers) from H to circles s, t, k are equal.

Thus the circles, s, t, k are coaxal, the radical axis being MH, and their centres O_1, O_2, K are collinear.

As KL is the polar of H for c, and therefore perpendicular to MH, the radical axis of s, t, it follows that L also lies on O_1O_2 and the common chord KL of the pedal circles of A and D passes through the mid-points O_1, O_2 of BH_1, CH_2 as was to be proved.*

If K is inside the circle c, a similar argument can be applied with K and L interchanged.

The above figure also leads to a solution of another problem arising out of the articles mentioned.

The point O, which is the other meet, besides L, of the pedal circles of A and D, is shown to lie on the other two pedal circles and also on the four nine-point circles of the triangles formed by taking three of the points A, B, C, D.

The point O, which is also the centre of the rectangular hyperbola h, also lies on the circumcircle of the diagonal triangle UVW of the quadrangle $ABCD$, a fact that prompted Mr. Lawlor to suggest that O should be called the *nine-circle point* of $ABCD$.

For a proof of this last property the aid of the rectangular hyperbola h was called in. In Note 1933, *Math. Gazette*, Oct. 1946, *A Geometrical Query*, the present writer asked for a proof of the property using only circle theorems. This problem proved far from easy, and the appended solution is based on results obtained independently by Mr. W. J. Hodgetts and Mr. T. McHugh.

Let $\quad AD,\ BC = U, \quad BD,\ CA = V, \quad CD,\ AB = W.$

Then by Pascal's theorem applied to $ARSDQP$, we have VW contains H, which is the inverse with respect to c of the point O, the other meet, besides M, of the nine-point circles MRS, MPQ of the triangles ADC, ABD; for these circles invert with respect to c into RS and PQ.

Since U and VW divide AD harmonically the inverse of U for c also lies on VW.

Since two pairs of inverse points are concyclic,

$$\angle(OU, MO) = \angle(MU, VW), \qquad \text{.....(1)}$$

where (a, b) is the *directed angle* (cross) between the lines a and b.

Similarly if N is the mid-point of AB we have

$$\angle(NW, UV) = \angle(OW, NO) \qquad \text{.....(2)}$$

* AQ, DP meet at H_1 and AS, DR meet at H_2. By applying Pascal's theorem to the hexagons $APSDQR$, $AQRDPS$, $AQSDPR$, and $ASQDRP$ it is seen that K, L are diagonal points of the quadrangle BH_1H_2C and hence KL bisects the parallel sides BH_1, CH_2.

But, if Z is the mid-point of BD, the point O lies on the circle MNZ, which is the nine-point circle of ABD, and therefore

$$\angle(MO, MZ) = \angle(NO, NZ)$$

or, by parallels,

$$\angle(MO, NW) = \angle(NO, MU) \qquad \text{.....(3)}$$

From (1), (2), (3) follows, by addition,

$$\angle(OU, UV) = \angle(VW, WO).$$

Hence O, U, V, W are concyclic.

COROLLARY. If A_1, B_1, C_1, D_1 are the centres of the pedal circles of A, B, C, D then the quadrangles $A_1B_1C_1D_1$ and $ABCD$ are indirectly similar.

For the point O is the inverse of H with respect to the circle c, and O, H, M are collinear.

A_1D_1 is perpendicular to KL, the common chord of the pedal circles, and therefore parallel to MH, i.e. to OM, since H and KL are pole and polar for c.

Similarly A_1B_1 is parallel to ON.

Since O lies on the circle MNZ, the nine-point circle of ABD,

$$\begin{aligned} \angle D_1A_1B_1 &= \angle MON, \\ &= \angle MZN, \\ &= \angle BAD, \text{ by parallels.} \end{aligned}$$

A similar result follows for the angles between any corresponding pairs of sides of the quadrangles, which are therefore indirectly similar.

Returning to P. W. Wood's article, quoted above, several results were obtained and quoted about the *isoptic point* of $ABCD$, which we call here E to avoid confusion. E is the common point of the circles of similitude of pairs of the circles ABC, BCD, CDA and DAB.

An additional property of the isoptic point has been pointed out by Mr. T. McHugh, who also supplied a simple geometrical proof:

Any two of four coplanar points A, B, C, D subtend at their isoptic point E an angle which is equal to the sum or the difference between the angles which the two points subtend at the remaining two points, e.g.

$$\angle AEB = \angle ACB + \angle ADB.$$

for the points A, B, with similar results for any other pair from A, B, C, D.

The use of directed angles avoids any ambiguity in the sign.

It may be observed that if A, B, C, D are concyclic the above definition of the isoptic point is not clear, whereas the property given above shows that E is at the centre of the circle $ABCD$.

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MATHEMATICS TEACHING IN RUSSIAN SECONDARY SCHOOLS

By W. O. Storer

The purpose of this article is twofold. In the first place it aims to give a general picture of the content and method of school mathematics teaching in a country in whose scientific and technical education the Western world is becoming increasingly interested. This general picture is here based entirely on publications of the Ministry of Education of the RSFSR (Russian Soviet Federal Socialist Republic) and on text- and other books in general use, mostly from the Russian Republic but a few from the Ukraine.* It is not based on personal contact with the system or with teachers who have worked within it, but it is hoped to extend this article at a later date on the basis of first-hand experience. Aware of the diversity and flexibility of our own system of mathematics teaching, we would not expect a foreigner to understand it fully from publications alone. The Russian System, however, is less extensive and more unified, and we may therefore hope to obtain a useful understanding of it from books alone. We must remember, though, that it is in a process of change, and also that, as in England, there will be considerable variability from one school to another depending on the quality of the teachers. However oversimplified this introductory article may be, it should at least serve to give an orientation towards the subject.

The second aim of the article is to let some of the differences between the Russian and the English systems prompt an examination of some aspects of our own system. Some characteristics of our own methods in school mathematics which strike foreign teachers as strange are hardly observable by us except in contrast to differences in other countries. If such characteristics are to be retained it is desirable that they should be consciouly understood and deliberately adopted and not just traditionally followed.

The Russian school system begins with a four-year primary school for the ages of 7 to 11, following optional attendance at kindergarten schools. The complete secondary school course is then for six years, from 11 to 17, but there are also incomplete secondary schools where the course is followed only for three years, until the age of 14. From the incomplete secondary schools children go to a wide variety of vocational and trade schools and to training colleges for kindergarten and primary school teachers. The mathematics teaching here considered will be that in the complete secondary schools, where a fraction of the population is educated that is comparable to the fraction educated in English secondary grammar schools. All pupils in these schools follow the same curriculum, and have to repeat a year's work if they do not reach the necessary standard. There is no specialization except in the form of extra-curricular societies.

A general view of the content of the course is most easily obtained from the official syllabuses (1) and a handbook such as (3). A fuller view of current ideas and activities is obtainable from the two-monthly official journal "Mathematics in Schools" (2).

Changes in the syllabuses were introduced in 1954, and these changes had spread as far as class VIII by the school year 1956–57. During that year

* A list of the books principally used in the preparation of this article is given at the end, with titles translated into English. The library of the Society for Cultural Relations with the USSR, 14, Kensington Square, London, W.8, contains a small mathematics section, and Collett's Russian Bookshop, 45, Museum Street, London, W.C.1, is another source. Gratitude is expressed to Dr. M. L. Cartwright for the loan of some of these books, and also for suggesting the writing of this article.

classes IX and X followed the old syllabuses. The direction of these changes is that which was followed in this country earlier in the present century. The old, rather narrow thorough study is being broadened and liberalised under the stimulus of the official insistence on a "polytechnic" education. Especial emphasis is being given to lines of development that have long been familiar to us. These include the importance of practical activities such as model-making, surveying, the use of diagrams and graphs, and mathematical tables; and the linking of mathematics with related subjects such as physics, chemistry, drawing and geography. Trigonometry is being introduced at an earlier stage, in class VIII (age (14–15), and the concept of function and its graphical representation are coming to be developed from the beginning of the course. Some relaxation is taking place in the traditional deductive method of teaching. In the lower three classes (V to VIII) instruction is no longer given in a purely deductive form but contains intuitive and visual elements, while in the upper three classes (VIII to X), although the method remains deductive, room is found for a visual or intuitive approach.

The official syllabus stresses that primary attention should be given to the fundamental mathematical concepts and ideas, and that the pupils' memory should not be overloaded with a large quantity of formulae and principles of secondary importance. Complicated and awkward transformations and problems are to be avoided. Revision of work is not to be merely repetition but a setting of the work within a wider context, with the appreciation of resemblance and differences between different sections.

These directives indicate clearly that changes are in progress. The abler teachers are encouraged to experiment with new methods and activities, and the less able ones are urged to follow the official lead. Experiments are reported and discussed and criticized, either at conferences, in journals or in published volumes. Considerable thought and activity are in evidence, and an atmosphere of constructive criticism prevails. In thus planning for change the ablest teachers and administrators yet show themselves aware of the danger of moving too far away from their tradition. For example, in their modification of the traditional deductive method mentioned above they express awareness of the danger of going to the other extreme of an "anti-deductive" method of exposition. Or again, the tendency to broaden the syllabus must not be allowed to go so far as to reduce the depth of understanding and conscious mastery of the basic mathematical principles.

To give definiteness to our discussion a brief indication of the content of the syllabus for each school year will now be given. Details will not be given when the content seems close to what we, with our tradition, would expect. In each year 6 hours a week for 33 weeks are allotted to mathematics, and three hours for homework, and the apportionment of this time between the different branches is given in the table shown on p. 23.

Arithmetic

Class V. Of the mathematical subjects only arithmetic is studied in this first year in the secondary school. The primary school work on numeration and calculation with integers is revised in conjunction with practice with the Russian abacus (which is still used in commerce). The laws and rules of the arithmetic operations. Factorization and related topics. The theory of vulgar and decimal fractions. Problems, including areas and volumes, and the circle and cylinder. Introduction to percentage and ratio. A little work with diagrams and practical geometry.

Class VI. In this class one-third of the time is devoted to arithmetic. Percentage is treated more fully, and the diagrammatic representation of data. Direct and inverse proportion.

There is no further study of arithmetic as an independent subject after this year. In the higher classes, however, teachers are required to give attention to the continued improvement of their pupils' habits in arithmetic by discussing the best methods of preforming calculations that arise in problems in other branches of the subject.

Class	Arithmetic	Algebra	Geometry	Trigonometry
V	6	—	—	—
VI	2	2	2	—
VII	—	4	2	—
VIII	—	4 for first half-year, 3 for second half-year	2 for first half-year, 3 for second half-year	—
IX	—	2	2 for first 3/4 year, 3 for last 1/4	2 for first 3/4 year, 1 for last 1/4
X	—	2	2 for first half-year, 1 for second half-year	2 for first half-year, 3 for second half-year

Algebra

Class VI. The usual basic work as far as simple linear equations with one unknown, simple graphs, and the multiplication formulae as far as $(a \pm b)^3$ and $(a^3 \pm b^3)$.

Class VII. Factorization and the division of a polynomial by a polynomial. In teaching the algorithm for the latter it is not now required to give a strict theoretical justification. In the study of algebraic fractions it is emphasized that in a given expression only certain values of the letters are admissible for the expression to have a meaning. The algebraic solution of a system of linear equations is based on the process of transforming the system into an equivalent system. The graphical illustrations of the three cases of intersecting, parallel and coincident lines. Simple inequalities.

Class VIII. Powers and roots; the extension of indices to the field of rational numbers. It is shown that the measurement of line-segments leads necessarily to the concept of an irrational number. This theory goes as far as showing that the decimal fraction representing an irrational number cannot be recurring, and strictly defines the operations on irrational numbers. Quadratic equations and the basic properties of quadratic functions, including their graphs. Simple systems of equations of the second degree are solved, with their graphical interpretation.

Class IX. For this and the following class the old syllabus was still in force for the year 1956–57. Sequences and limits are studied as a necessary preparation for the topic of the length of the circumference of the circle in geometry. In this work the theorems are in general formulated without proof, the emphasis being on the pupils' understanding of the basic concepts. The results are applied to the summation of an infinite decreasing geometric

series and to the conversion of a recurring decimal to a vulgar fraction. The theory of logarithms and the logarithmic curve. Index and logarithmic equations. The slide rule.

Class X. Complex numbers. Further work on inequalities. The "investigation of equations" here systematizes the ideas already discussed in classes VII and VIII.

Geometry

Class VI. Basic concepts, parallelism, triangles. Parallels are now treated before triangles in order to simplify the study of the latter. By the end of the year simple problems for proof are included, and the nature of a geometrical proof is taught, with the concepts of definition, axiom and theorem. Some practical work is included.

Class VII. The usual development of the subject with quadrilaterals and the circle.

Class VIII. The theory of the ratio and proportion of segments includes a strict discussion of commensurability and incommensurability and recurring and non-recurring decimals. The theory of irrational numbers is further developed in the algebra course. The theory of similarity is based on the transformation of homothesis, and the metrical relations of the right-angled triangle and the circle are taught, including the introduction of the trigonometric functions.

Class IX. The trigonometric functions introduced in class VIII are extended successively to the general angle. The radian measure of arcs and angles. Reduction formulae for the trigonometric functions of the general angle, their periodicity and graphs. Formulae for the angle-sum and for the conversion of a sum into a product. The use of 4-figure logarithmic tables of the trigonometric functions, and the solution of right-angled triangles.

Class X. The solution of oblique-angled triangles. The inverse functions, regarded as many-valued but having a principal value. Trigonometric equations.

Comments on the syllabuses

As is to be expected, these syllabuses follow the main lines of development of the subjects that are familiar to us from our school course, but there are some differences in timing and depth of treatment that will now be discussed.

Arithmetic

The content is close to that of the lowest two forms in our secondary schools, though it seems that it may be rather more concentrated, in spite of the fact that the full 6 hours a week are devoted to it. For one thing, English children have had more practice in their junior schools with conversion from one measure to another. For another, several theoretical points are studied more deeply in Russian schools than in most of ours. The difference is most strikingly evident from such a standard text-book as Kiselev (4). In his preface to the 1952 revision the reviser makes quite clear that he intends the book to be a strict development of the theory of whole numbers and fractions, and equally clear that the book is not intended to give practice in the application of the theory to problems. For this latter purpose one must turn to specially compiled books of problems, of which there are many in general use. (In contrast to this, text-books in other branches of the subject contain sets of exercises). The strict development involves difficult arguments that we would regard as fully of sixth form standard. The reviser prints them in smaller type, and recognizes that pupils will have to return to them later, but insists that

teachers should be fully conversant with them. The book, however, is not called a teachers' handbook but a text-book for classes V and VI, and to 11- or 12-year-old children it must seem rather severe. Probably it expresses an ideal, and most teachers do not expect too much from their pupils in the understanding of the general theory.

Algebra

The content of this course is close to ours, but in a few places the treatment is deeper, when we remember that this course is for the whole of the secondary school population and not only for the abler mathematicians. This is particularly noticeable in the treatment of irrationals and in the solution and "investigation" of equations. Equations are solved by transformation into an equivalent system. (Substitution is a particular case of this.) The criterion of equivalence is strict, by which, for example, the equations $x^2 - 5x + 6 = 0$ and $\frac{1}{x} + x^2 - 5x + 6 = \frac{1}{x}$ are not equivalent because they do not have the same set of admissible values of x. The expressions on the two sides of the second equation have no meaning for the value $x = 0$. In class X the topic of the "investigation" of equations arising from problems goes further into the equivalence of systems of equations and the appearance of extraneous roots. It also examines the sets of values of the variables and parameters that are admissible in the equations and also in the meaning of the problem, from which those roots can be chosen that are significant. The thoroughness of the treatment reminds one of the discussion in De Morgan's "Elements of Algebra" (1828) on the four operations with signed numbers, to show their unifying power in the solution of his "problem of the couriers". In comparison with this, the practice in English text-books of illustrating the rules for the multiplication of directed numbers by a mere two-page discussion on the motion of a train or uniformly changing temperature seems pathetically inadequate for developing the concepts and their mode of applicability. The fulness of treatment that is advocated by Russian writers should lead to an understanding that the routine symbolic transformations of algebra must be supplemented by conscious thought about their limitations of applicability. Discussions and book-reviews suggest, however, that this thoroughness, here again, is also an ideal rather than a present achievement. A recently published text-book was severely criticized in "Mathematics in Schools" for its loose treatment of the equivalence of equations, and the author was accused of not striving for the pupils' conscious mastery of the theory but only for a half-intuitive, half-dogmatic teaching of a technique.

Geometry

This subject is called at the outset the study of the spatial properties of the material world, and with this definition it is not surprising that the method of superposition is widely used in proofs of congruence, and other forms of movement and folding. (At the end of the course, however, an axiomatic system of Euclidean geometry is discussed, and the principle of superposition is related to it.) The content of the elementary work differs little from our own, as is to be expected. The distinction between a theorem and its converse is emphasized, and although examples given in discussion indicate that pupils often make the same confusion of thought that ours do, their teachers appear to be less easy-going in the matter. More striking differences appear, however, in the fullness of their treatment of irrational segments, the calculation of the circumference of the circle firmly based on the limit of a sequence of regular polygons, and the definition of area as an additive function. The same rigour

is to be found in the treatment of the areas and volumes of polyhedra and the cylinder, cone and sphere.

In teachers articles and discussions the same movement is evident as in this country to treat a geometrical problem not simply as a static situation for solution but as a functional variation of one element in response to the variation of other elements. This approach is encouraged in the lower as well as in the upper classes.

Trigonometry

There is nothing requiring comment in this subject except its slightly later start (class VIII, age 14–15) than is becoming customary with us.

The spirit of lively interest in the subject is to be sensed in the official journal "Mathematics in School" (2), published volumes of teachers' researches and studies such as "The Solution of Problems in the Secondary School" (8) and popular books for adult reading such as "Interesting Algebra" (10) and "Mechanical Oscillations" (11). There is inevitable conflict between the tendency to broaden the content of the school course along the lines of "polytechnic" education, and the traditional standards of rigour, for no extra time is to be given to the subject. The official directive is that although there may be some reduction in rigour the aim of the teaching is to be conscious understanding and not just the ability to follow out dogmatic methods automatically. Because of the limitation of teaching time it is recognized that to try to extend the syllabus at the upper end (e.g. with analytical geometry) would involve loss in depth elsewhere. At the lower end, problem material is taken increasingly from social and economic data for the country, but one reviewer has protested that if this trend is pushed too far the pupils of class V will need almost a course in modern history and political economy in preparation for their arithmetical problems. Commonsense comments such as this lead one to the conviction that a sound balance will be found between the new trends and the old tradition.

"Mathematics in School" contains articles at different levels and of different natures. Some are clearly designed to help the rank and file teachers who had been trained to teach a different syllabus to carry out the modern requirements, e.g. in graphs and other work of a diagrammatic and practical nature. We are probably surprised to find it necessary to give so much attention to such elementary ideas. Other articles concern teaching-method at higher levels, or report experiments by teachers either in formal lessons or in extra-curricular work such as "excursions" or "mathematical clubs". These latter provide the only extensions of the common syllabus for those pupils who show special ability in the subject. For example, one reviewer recently reported with some surprise that the elements of the differential calculus were being studied in some of these clubs. It seems likely that they will become the main agency for any gradual extension of the content of the school course. The journal also contains articles on historical topics, book reviews, reports of conferences and problems set and solved.

The articles in the symposium (8) are an indication of the attention given by the best teachers to methodological as well as to purely mathematical aspects of their work. The most interesting to us are those discussing methods of improving the pupils' solution of problems in arithmetic and algebra in the lower classes. Close consideration is given to both the logical and psychological steps in formulating a problem and reaching a solution. In places the detailed study gives the impression of going too far in attempting to establish a formalized procedure for the solution of problems, but much of the discussion is highly stimulating. Teachers deplore the poor performance of class V pupils in arithmetical problems involving speed and distance, rates of work,

taps and sinks, but we would probably take the view that they are expecting too much at that age, and that they would do better to extend their arithmetic course into the middle classes of the secondary school. A useful suggestion, however, is that the pupils should compile a mathematical dictionary to which they can refer, containing not only units, measures, rules of the operations, but also explicit statements of the functional relationships between related quantities. From the quantities occurring in the statement of a problem, reference to the dictionary will then give a reminder of the type of relationship to set up in one's solution. After solving an arithmetical problem children are trained to express the solution as a single formula containing only the numerical data and symbols of operation, in order to give a lead towards algebra later. In discussing the solution of algebraic problems, the important point is made that there are always two ways of setting up an equation that will lead to a solution, and teachers are urged to insist that their pupils investigate both lines of solution. Two of the unknowns entering into an elementary problem (e.g. speed, distance, time) are usually related with a known quantity by an equation either of the form $x = ky$ or of the form $xy = k$, where k is the known quantity. If x is taken as independent an expression can be obtained for y, using each item of data in turn. The required equation is then obtained by relating these various expressions for y in accordance with a further remaining item of data. But the two unknowns could interchange their roles, and expressions for x in terms of y be used instead to set up an equation. (E.g. for the problem: "A train takes $8\frac{1}{2}$ hours for a journey of 300 miles, travelling for part of the journey at a speed of 40 m.p.h. and for the rest at 30 m.p.h. Find the distance and time travelled at each speed" we obtain, with an obvious notation, the alternative equations $40x + 30(8\frac{1}{2} - x) = 300$ and $\frac{y}{40} + \frac{300 - y}{30} = 8\frac{1}{2}$.) It is highly instructive to use both methods and compare the solutions.

This example is cited as an illustration of the conscious attention that is being given to method and the emphasis on the pupils' understanding. This attention is, of course, also to be found in English writing on mathematics teaching, but at the present time it is not much brought to the notice of the practising teacher. Second only to the big disparity in the range of topics treated in the upper classes (because of the advanced specialist courses in English schools) this is the most striking contrast in the present-day school mathematical literature of the two countries. Very little is now being written in England on understanding at this level of elementary mathematics. Topics are discussed, neat mathematical methods and visual aids, all of which are valuable in improving the quality of teaching. Text-books provide increasingly good arrangements of the material and exercises, and occasionally introduce valuable new ideas. But fundamental understanding is little discussed. One reason for this may be that some of the ground was covered 20 to 30 years ago, notably in the earlier reports of our Association, and that in this country we are now at a later stage of development. Another reason may lie in the organization of teacher training and educational research in the two countries. In Russia teachers for the upper classes of secondary schools receive their education and training in the four-year course at an Educational Institute which has university status. These Institutes and the Academy of Pedagogical Sciences of the RSFSR are together likely to direct more deliberate attention to educational problems in the different special subjects than are our own more loosely coordinated agencies with less clearly defined functions in this field. The more limited school syllabus is also likely to concentrate the attention of Russian teachers more fully within a narrower range, whereas our own wider syllabus attracts the attention of the ablest teachers to the more advanced work.

While broadening their syllabuses the Russians show themselves aware of the danger of becoming superficial and relying on techniques and rote memory. We ourselves may already have gone a little too far in this direction, and the time may be ripe for a renewed attempt to find a satisfactory middle path. Our present quality of teaching is reasonably successful with the group of abler children who later specialize in the exact sciences, but its achievement with children of average grammar school ability is far below what is desirable. The problem is, no doubt, linked with the inadequate supply of competent teachers, and all countries suffer from this. It also seems to be linked, however, with the English characteristic of teaching the subject inductively, whereas on the Continent the teaching is predominantly deductive. Our inductive method, which admittedly has its place and value, often goes to the extreme of talking round and round the subject, hoping that thereby it will be understood, but omitting to complete the treatment by a deliberate attack on the central core, with the enunciation of strict definitions and principles.

This necessary reconsideration of what we should aim to achieve in understanding in the main course of the secondary school must, of course, be based on the pupil's stages of intellectual growth. English mathematics teachers have a storehouse of experience in this, but most of it has been no more explicitly formulated than much of their teaching. This formulation of the psychology of intellectual growth is, admittedly, not primarily the responsibility of teachers of mathematics, and its inevitable lack of precision makes it uncongenial to them. Psychologists, notably Piaget, have thrown valuable light on the intellectual development of younger children, but have not yet been able to extend it to the level of secondary school mathematics, perhaps because of its greater complexity and their own lower competence at this level. The Gestalt psychologists, Wertheimer for example, have suggestive things to say about teaching for understanding but few clearly established facts. We must look to mathematics teachers themselves for the necessary experience and interpreation of experience on which our teaching for understanding must be based. Only through this can we find the correct balance between the interest of a broader content of the school mathematics course and an understanding of the basic concepts and principles. Only with such a balance can the mathematical education of that large group of grammar school children who have average ability be improved significantly.

Note on Terminology and Notation

Finally, a few remarks are offered mainly on differences in terminology and notation between Russian and English usage. Most of the usages discussed will not in fact be distinctively Russian but Continental. They may suggest minor improvements for us.

Most teachers who have observed the appalling confusion between addition and multiplication among the mediocre and weaker pupils (partly caused by the teacher's neglect to discuss it explicitly) would be glad to remove the confusion between the visual symbols $+$ and $\times$. The latter symbol is little used for multiplication on the Continent, the dot . being used instead, even with numerals. For us to abolish $\times$ in favour of . would bring confusion with our decimal point, but if we further replaced the latter by the comma, we should have agreement with Continental usage and also the abandonment of $\times$. There is no correspondingly cogent reason for replacing our division sign $\div$ by the Continental : or fraction line — except for conformity. The Russian setting out of the long division algorithm, however, has the divisor on the right of the dividend, and this agreement with the order of $168 \div 14$ is advantageous.

Our own sixth-form work with logarithms would be helped by the general adoption in school textbooks of the notations $\lg = \log_{10}$ and $\ln = \log_e$, and

it is high time that our misleading notation $\sin^{-1}$ for the inverse trigonometrical functions was abandoned in favour of the general use of arcsin.

In geometry precision is encouraged by the fact that Russian has two related words, one for the circumference of a circle and the other for the interior. As in many other languages, the Russian word for "number" when used as a name or ordinal (e.g. "room number 7") is different from the usual word, and thereby calls attention to a difference in concept which is obscured in English. The greater richness of the Russian language in variety of participles, and their more frequent use as nouns, make less frightening and more intelligible those concepts for which our children can only use the strange terms addend, dividend, multiplicand.

There is one mode of formulation of the angle properties of the circle that is in general use on the Continent but little in evidence in this country, that gives both clearer insight and a valuable generalization. It is, that the angle in a circle is measured by half the arc on which it stands, and that the angle between a tangent and a chord through the point of contact is measured by half the arc included between them. From these, we obtain a generalization for the angle between a pair of chords, a pair of secants, a pair of tangents, and a tangent and a secant, in terms of the arcs cut off.

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1906. They pursued the logic of several incompatible opinions to absurd ends, without perceiving the incongruity. With cool head and tranquil judgement, imperturbably unconscious of the flight, they oscillated from asymptote to asymptote.*—T. E. Lawrence, *Seven Pillars of Wisdom* 1926. Chap. 3.

* The metaphor of oscillation 'from asymptote to asymptote' originated in a conversation with a friend who tells me he mistakenly applied the term "asymptote" to the branches of the hyperbola. (Footnote inserted by A. G. Lawrence in the standard edition of 1940). [Per Professor E. H. Neville.]

1907. The wind has blown away the fog except for occasional patches: these patches are nowhere dense.—B.B.C. Weather forecast, 11 p.m. Dec. 23rd 1956. [Per Mr. J. C. W. De la Bere.]

A ONE-YEAR NON-EXAMINATION SIXTH FORM COURSE IN MATHEMATICS

By Joyce E. Wilson

In outlining the following course planned for girls of sixteen to eighteen years of age, I should make it clear that it occupied only a subordinate place in school organisation. The orthodox General Certificate of education ordinary level course is taken by all girls in the school for the first five years (with the exception of perhaps two or three in the fifth year) and we have always had a strong group for Advanced level in the Sixth. Indeed in the four girls' schools which I have known intimately, the Mathematics course has always been highly popular and more successful (if judged by examination results) than most other school subjects. I do not believe—on such facts as I have—that there is a sex difference in mathematical ability. The lower total number of girls taking the General Certificate of Education examination is, I fancy, explained on other grounds.

Nevertheless in my own school we have from time to time to provide a mathematics course for:—

1. girls who have failed to keep up with the Ordinary level classes (perhaps two or three of these in any one year at fifth form level).
2. girls in Form VI who may or may not have secured General Certificate of Education Ordinary level who wish to continue their Mathematics until they go on to Training College but who could not cope with Advanced level.
3. girls who are transferred from Modern school at fifteen or sixteen who will achieve their five passes at Ordinary level in other subjects and go on to Training College but who have not the usual Grammar School background in Mathematics.
4. girls who are following a pre-nursing or large-scale catering course in Form VI.

With these girls I have set out with certain aims in view:–

A. To fight for greater facility in simple arithmetical calculations. All these girls (I.Qs $\leq$ 100) should reach the standard of the Arithmetic paper at Ordinary level in General Certificate of Education—at any rate dealing with straightforward questions involving only mechanical operations.

B. To try to induce pleasure in and respect for Mathematics as a whole in a very elementary way and to remove the panic that is sometimes a reflex action on any mention of the subject.

C. To insist on "proper" mathematics, not history, drawing or discussion based on a mathematical topic.

D. To produce a sense of individual achievement and a glimpse of the pattern and orderliness of the subject without involving the class in too much manipulation of symbols or deductive reasoning.

We started off with one great advantage—no examination clouding the horizon, no external or school test at the end. In point of fact we ended with quite a tangible record, a file of graphs, curve tracings, calculations etc. that several girls were obviously proud of and wanted to keep. Such a course can hardly succeed with less than three periods a week for a year and I demanded a good deal of homework. At no time was there any suggestion that this course was a 'soft option'. Sympathy and help were willingly given to the slow-witted but not to the lazy.

One period a week was given specifically to Arithmetic and a paper (generally G.C.E. 'O' level Arithmetic) done every other week for homework.

The first topic considered was the revision of very elementary operations. This included addition using scales other than 10 and an inquiry into three

different methods of subtraction. (The girls were old enough not to be confused or deflected from their own methods of a school-life-time). We spent at least one lesson assessing the advantages and disadvantages of the two common systems of weights and measures (1) the English system built up over the ages for the everyday convenience of that untidy and illogical individual T. C. MITS*, but causing irritation to school children and foreigners and (2) the Metric system as a mathematical consequence of the French Revolution, neat, tidy, logical, ideal in the abstract for the scientist and for international commerce but often inconvenient for T. C. MITS who invariably needs a little bit of paper to work out his decimals! "Companion to Elementary School Mathematics" F. C. Boon, chapter 8 has some useful information. We considered all sorts of devices to save labour; I was asked to show the large scale catering group the use of ready reckoners as these were considered a 'must' for the costing of school meals; alas, when we produced our ready reckoners in class, I was the one who knew least how to use them! We removed the word 'approximate' from our vocabulary and tackled 'degree of accuracy'. We used slide-rules and I wanted the girls to make one. I had plenty of logarithmic graph paper but never got down to practical operations. Is this sort of thing possible? Can logarithmic graph paper be used in other simple ways? I have a quantity of logarithmic graph paper left on my hands. Logarithms are valueless unless applicable to all numbers including those less than unity and of course we ran into trouble with the negative characteristic. The revision of directed numbers led me to embark on some graphs of trigonometrical ratios and eventually to attempt polar coordinates.

All the class had met the sine and cosine in the first quadrant, some had even dealt with obtuse angles but none had gone further, nor had any of them an idea of the continuous nature of the graphs of these functions. It took us a long time to determine the sine of angles 10°, 20°, 30°, ... 360°, tabulating every step. It was entertaining to see the varying speeds with which the girls discovered that it was unnecessary to look up tables after 90° and that it was not necessary *each time* to draw a figure to determine the sign. On graph paper drawn in inches and tenths we used 0·5 to the inch for the ordinate and 50° to the inch along the abscissa; we plotted points up to 720°. The determination of the values at 0°, 90°, 180° etc. took some time and there were some raised eyebrows at our conclusions: "it must be 1 (or 0) *because it fits in with the shape of the curve*". The girls tabulated the cosine data and drew the curve on the same paper without any further help from me. Again I enjoyed the moment when each girl realised the similarity in the graphs and the labour that that recognition could save her. The resulting graphs certainly inspired a sense of achievement and were proudly displayed on the form room walls for the admiration of more orthodox mathematicians.

It was a short step from this to polar coordinates which gave another fairly simple illustration of directed numbers. Polar graph paper is essential and can be obtained from Heffers at Cambridge. Tabulation again was a slow process but justified by results. I took the idea and practical hints from Cundy and Rollett's "Mathematical Models" page 67 and restricted operations to the simple rose-curves $r = 10 \cos 2\theta$ and $r = 10 \cos 3\theta$ and the cardioid. We started on three petals; $r = 10 \cos 2\theta$ was done for homework and not everybody I am sorry to say produced four petals. Some jumped to the conclusion that there would be two and cooked calculations accordingly, only to be confounded at the subsequent lesson. The cardioid was easy to calculate, pleasantly different in shape and we met it again later in another guise. While we were doing this work it gave me great pleasure to warn my girls that it was useless to try to "get help" from boy friends in the adjacent

* *The Common Man in the Street* (Lieber and Lieber)

grammar school as even the Advanced level groups would not include this in their syllabus. Confidence increased by leaps and bounds!

The rose-curves led to a discussion of families of curves and family likenesses. A respite from graph paper and tabulation was obviously needed, so plain foolscap paper became our medium (a great rest to the eyes) and we constructed two sets of curves, conchoids and limaçons, by simple drawing and measurement. The details I took from Mr. Boon's "Companion to Elementary School Mathematics" pages 81 and 82. On plain foolscap paper we took the fixed point two inches from the fixed line; then with the points on the movable straight edge $3\frac{1}{2}''$ apart we got a good sized loop, $2''$ apart a cusp, and $1\frac{1}{2}''$ apart the curve flattened further. These three curves can be conveniently drawn on one half of the sheet. Then on the other half of the paper can be constructed a limaçon with loop, a cardioid and a limaçon without loop. [Substitute for the fixed line a circle radius $\frac{3}{4}''$ and mark the fixed point on the circumference; on your movable straight edge mark points (a) $1''$, (b) $1\frac{1}{2}''$ (c) $2''$ apart. These measurements give suitably sized curves for foolscap paper and the total effect, if carefully drawn, is pleasing.]

Graphs and the graphical method served us as a topic for months. We amassed a remarkable collection of printed graphs from the newspapers. One I remember recorded the volume of sound at a football match. The goals were easily spotted but we noted with British approval that the highest volume of applause was recorded when two policemen collided. The sort of graphical work for which data is provided in text books has always seemed to me rather tedious and unconvincing. So much better to collect one's own data! For this some of the questions from the Cambridge 'O' level Practical Physics examination are quite excellent, particularly those on mechanics topics—swing of pendulum, law of the lever, rate of flow from a burette etc. Little knowledge of theory is required and the laboratory operations are simple. The ensuing graphs and the use of these graphs to obtain results is the interesting part. I, quite unashamedly, used the experiments to emphasize the graph. In most schools (mine is no exception) laboratory space is limited and the first essential step is to secure the sympathy of the Head of the Physics department.

Of course little bits of history crept in. Whatever course I am taking with whatever age group certain names for example inevitably intrude; Archimedes (and it is a crime to allow him only to be associated with a bath), Galileo (and less dramatic but equally convincing experiments on the acceleration due to gravity can be done in the form room), Leonardo da Vinci (one of us was inspired in the course of a summer holiday to chase across Munich from picture gallery to picture gallery to seek the little Leonardo da Vinci "Madonna mit Kinde"). With Sir Isaac Newton there is the temptation to insist on his "greatness" while recounting only stories of his absentmindedness. The falling apple* must be accompanied by mm'/d^2 and next time I take this course I am going to spend some time on "First ideas of the Calculus". Gradients would fit in nicely with our other graphical work and I can see no reason why the girls should not be able to understand at least velocity from the space-time curve. After Newton, the mathematical work becomes practically incomprehensible to such a class and names are correspondingly meaningless. Nevertheless Descartes turned up quite profitably and, with no excuse whatsoever, I like to mention Faraday the non-mathematician, as the father of the now very mathematical study of electricity. In any case Faraday is worth knowing as a man. My girls know fairly well what to use as a red herring at what time. I allow myself to be drawn and each side is under the impression of controlling the situation. Honour is satisfied on both sides and

* *See* Otto Wili Gail's *Romping through Physics*, pp. 38–40.

a good time is had by all. We do *not* officially study the history of Mathematics. In recent years a number of good biographies suitable for children have appeared, e.g. Andrade's, "Sir Isaac Newton." Lewis's "Brief Lives. Leonardo the Inventor." Cottler's "Man with Wings." Kendal's "Life of Faraday."

In conclusion let me admit freely that I have devised nothing new or original—I have picked other people's brains for ideas and shall continue to do so. I should welcome suggestions for my next course from readers who have doubtless experimented in other directions more successfully. Can I assess the value of my own course? No. Did I achieve any of my aims? I do not know. My pupils are very polite. Of one thing only can I be quite certain. I personally enjoyed the teaching.

J. E. W.

High School, Gainsborough, Lincs.

CORRESPONDENCE

THIS TYRANNY OF DEGREES

To the Editor of the *Mathematical Gazette*

Dear Sir, $(\pi/2)^2 = (90°)^2 = 8100$. Yes, this is what we have come to now. It was written by a Sixth Form boy in an A & S Level paper. He had evaluated a definite integral without many mistakes, and having arrived at the rather too high-brow-looking result $(\pi/2)^2$, decided to express it in plain numbers that any schoolboy can understand. So $(\pi/2)^2 = (90°)^2 = 8100$. Answer.

Why? Because $\pi/2$ is nothing but a faddy schoolmaster's name for 90°: so let's be practical.

For fifty years I have been watching with growing dismay the ever-increasing tendency of boys (yes, and some older folk too) to think of angles exclusively in terms of degrees. The result on their minds has been disastrous. Ask some boys, if you dare, what an angle is, and note how many answer "Degrees". By now I am afraid to ask a boy the meaning of a right angle: I would be too likely to burst a blood-vessel when he inevitably answers "90 degrees". He evidently thinks that in the beginning God created degrees, and men came along afterwards and build up right angles out of them. The truth is the exact opposite of this: what God has done for you is to put you into a world full of right angles, and you were familiar with the look of them before you had learnt to suck a bottle. The Babylonians, Adolf Hitler and others have devised various ways (and not very good ones) of dividing the right angle, and boys have enthroned these divisions as divinely appointed units.

Here are some concrete instances for you: you can test them for yourself. Question in an exam paper: "How many right angles do the angles of a triangle make? Give a short reason." The most frequent answer. "Two, because that makes 180°". "Two angles of a triangle are 5/11 and 7/11 of a right angle. What fraction of a right angle is the other?" Nearly every boy who tried it reduced the angles to degrees, and back again to a fraction of a right angle after the subtraction, though of course most of them did it wrong.

Can stupidity go further? But was it really stupidity, or just corruption of the mind by vicious teaching? Is this perhaps the natural way for a boy who has never heard of a right angle except by the name of "90°", or worse still just "90"?

Writing last March in one of our most respectable papers, a celebrated sportsman attributed the excellence of the Oxford crew, then hot favourites, to their application of the maximum effort when the oars were at ninety degrees to the boat. After that Cambridge won.

Nobody wants to abolish degrees: in their right place they are harmless enough, though 60ths of a right angle would have been a better unit to choose but for astronomical and theological complications in ancient Babylon. But this monstrous supremacy of the degree over Nature's right angles and radians must be broken if sanity is to survive. It begins with the masters, who are more to blame that the boys, sloppily calling a right angle "90"; and it reaches its climax in "$(\pi/2)^2 = (90°)^2 = 8100$". Who will join in a firm stand against the usurper?

Yours etc., W. HOPE-JONES

To the Editor of the *Mathematical Gazette*

DEAR SIR,—Prof. Watson has very kindly drawn my attention to a geometrical proof by J. W. L. Glaisher of the identity

$$\sum_1^N n^3 = \left(\sum_1^N n\right)^2,$$

which is the subject of the first part of my note on sums of powers of the natural numbers ("Mathematical Gazette", October 1957, p. 187). Glaisher's proof may be familiar to many readers of the "Gazette", but was new to me; it is given in "Messenger of Mathematics", III (1874), p. 5. It is equivalent to mine, though it looks different because it is expressed in geometrical language.

Suppose we take two axes at right angles, intersecting at O. Given the sequences a_n, b_n, we mark off in succession lengths $OX_1 = a_1$, $X_1X_2 = a_2$, ..., $X_{n-1}X_n = a_n$, ... on the first axis, and $OY_1 = b_1$, ..., $Y_{n-1}Y_n = b_n$, ... on the second. We then complete the rectangle R_n with OX_n, OY_n as sides; the lengths of these are A_n, B_n, where $A_n = a_1 + a_2 + \ldots + a_n$, $B_n = b_1 + b_2 + \ldots + b_n$, and so the area of the L-shaped region between R_n and R_{n-1} is $a_nB_n + b_nA_{n-1}$. From this point of view, the process of partial summation used in my note simply expresses the fact that the area of R_N is the sum of the areas of such regions for $n < N$ (taking R_0 as having zero area). Glaisher's proof consists in applying this geometrical idea to the special case $a_n = b_n = n$ (the areas being evaluated geometrically by induction). He also points out that the same idea can be used to demonstrate the formula for $\sum_1^N n$, and his version of this, using rectangles of sides n, $n + 1$, is a little neater than mine. Thus, apart from certain differences of detail, my proof may be regarded as a translation of Glaisher's into the language of analysis.

Yours, etc., S. M. EDMONDS

Newnham College, Cambridge

MATHEMATICAL NOTES

2739. Generalisations of Schur's inequality

Schur's inequality and its generalization by Wright (note 2627) are included in the following inequality:

Suppose that

$$x > y > z \quad \text{or} \quad z > y > x, \qquad \ldots\ldots(1)$$

$$v^{1/2} \leq u^{1/2} + w^{1/2} \qquad (u > 0, v > 0, w > 0). \qquad \ldots\ldots(2)$$

Then

$$u(x-y)(x-z) + v(y-z)(y-x) + w(z-x)(z-y) \geq 0. \qquad \ldots\ldots(3)$$

The sign of equality is needed if and only if equality holds in (2) *and also*

$$y - z : x - z : x - y = u^{1/2} : v^{1/2} : w^{1/2}.$$

Proof. Clearly (3) holds if and only if

$$v \leq u\frac{x-z}{y-z} + w\frac{x-z}{x-y}; \qquad \ldots\ldots(4)$$

the function of y on the right (y between x and z) reaches a minimum where

$$\frac{u}{(y-z)^2} = \frac{w}{(x-y)^2}, \quad \frac{u^{1/2}}{y-z} = \frac{w^{1/2}}{x-y} = \frac{u^{1/2}+w^{1/2}}{x-z}; \qquad \ldots\ldots(5)$$

and its minimum is clearly $(u^{1/2} + w^{1/2})^2$. The inequality follows at once.

2. More generally let n be any real number, $n \neq -1$: let $f(t) = t^n$ $(t > 0)$, $f(t) = -f(-t)$ $(t < 0)$.

Suppose that

$$x > y > z \quad \text{or} \quad z > y > x, \qquad \ldots\ldots(6)$$

$$v^{1/m} \leq u^{1/m} + w^{1/m} \quad (m = n+1) \quad (u > 0, v > 0, w > 0). \qquad \ldots\ldots(7)$$

Then

$$uf(x-y)f(x-z) + vf(y-z)f(y-x) + wf(z-x)f(z-y) \geq 0 \ldots\ldots(8)$$

Equality occurs in (8) *if and only if equality occurs in* (7) *and also*

$$y - z : x - z : x - y = u^{1/m} : v^{1/m} : w^{1/m}. \qquad \ldots\ldots(9)$$

Without loss of generality let $x > y > z$. Let

$$F(y) = u\frac{(x-z)^n}{(y-z)^n} + w\frac{(x-z)^n}{(x-y)^n} \quad (n \neq -1).$$

$F(y)$ reaches a minimum M where

$$\frac{u}{(y-z)^m} = \frac{w}{(x-y)^m}, \quad \frac{u^{1/m}}{y-z} = \frac{w^{1/m}}{x-y} = \frac{u^{1/m}+w^{1/m}}{x-z}.$$

Plainly $$M = (u^{1/m} + w^{1/m})^m.$$

If $v \leq M$, then $v \leq F(y)$ $(x > y > z)$ and the inequality with the gloss follows at once.

Of course, if $n = -1$, (7) is to be replaced by $v \leq \text{Min}\,(u, w)$.

Plainly (7) is satisfied if v lies between u and w (in the wide sense, so that (8) or (3) hold strictly when y lies strictly between x and z and v lies between u and w.

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2740. Schur's inequality

In recent issues of the Gazette three notes have appeared which deal with the inequality

$$x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0 \quad \ldots\ldots(1)$$

where x, y, z are positive.

There is an interesting geometrical problem which leads to this inequality. Let C be a quadric cone. If its three principal planes are taken as co-ordinate planes, its equation is

$$ax^2 + by^2 + cz^2 = 0. \quad \ldots\ldots(2)$$

Now it is easily shown that, if C has a set of three mutually perpendicular generators, then

$$a + b + c = 0. \quad \ldots\ldots(3)$$

If the geometry is complex, it may be shown that, conversely, if (3) holds, there are sets of three mutually perpendicular generators, and one of the generators may be chosen arbitrarily, so there are infinitely many sets.

We now ask whether the result is still true in real geometry, that is, if a, b, c are real, and a real generator is chosen arbitrarily, are there two other real generators which make up a set of three mutually perpendicular generators?

If the direction ratios of the chosen generator are $l : m : n$, then

$$al^2 + bm^2 + cn^2 = 0. \quad \ldots\ldots(4)$$

The other two generators lie along the intersection of the cone (2) with the plane

$$lx + my + nz = 0. \quad \ldots\ldots(5)$$

From (2) and (5), we have

$$a(my + nz)^2 + l^2(by^2 + cz^2) = 0,$$

that is,
$$(am^2 + bl^2)y^2 + 2amnyz + (an^2 + cl^2)z^2 = 0.$$

The condition that the values of y/z should be real is

$$a^2m^2n^2 - (am^2 + bl^2)(an^2 + cl^2) \geq 0,$$

that is,
$$-(bcl^2 + cam^2 + abn^2) \geq 0. \quad \ldots\ldots(6)$$

From (3) and (4), we have

$$a : b : c = m^2 - n^2 : n^2 - l^2 : l^2 - m^2,$$

so (6) becomes

$$\Sigma l^2(l^2 - m^2)(l^2 - n^2) \geq 0,$$

which is equivalent to (1). The result is therefore true in real geometry.

It may be added that there is a much simpler proof of this result. For if the direction ratios $l_1 : m_1 : n_1$ and $l_2 : m_2 : n_2$ of the other two generators were not real, they would clearly be conjugate complex (that is, l_2, m_2, n_2 would be proportional to the conjugates of l_1, m_1, n_1). Since the product of two conjugate complex numbers is positive, the condition for perpendicularity

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

would not be fulfilled.

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2741. Football probabilities

1. During the broadcast of the draw for the 16 matches in the fourth round of the F.A. Cup (1957), the commentator, Mr. Raymond Glendenning, said he thought that, since 6 of the teams were from London,* there was a good chance of a match between 2 London teams. It is interesting to investigate what the chance is. We consider first the general case, where there are $2n$ teams, of which m are from London.†

Throughout we take account of the order of the n pairs of opposing teams, but we ignore the order in each pair, that is, we ignore the distinction between home and away teams. The total number of ways in which the $2n$ teams can be paired off is $(2n)!/2^n$.

We now calculate the number of ways in which the draw can be made so that no 2 London teams play each other. The number of ways of placing the m London teams in the n pairs, with no 2 in one pair, is $n!/(n-m)!$, the number of ways of choosing m of the $2n-m$ non-London teams to play against them is $(2n-m)!/(2n-2m)!$, and the number of ways of pairing off the other $2n-2m$ non-London teams is $(2n-2m)!/2^{n-m}$. Hence the number of ways is $n!(2n-m)!/(n-m)!2^{n-m}$. The probability that no 2 London teams play each other is, therefore,

$$\frac{n!(2n-m)!2^m}{(n-m)!(2n)!},$$

that is,

$$\frac{(2n-m)(2n-m-2)\dots(2n-2m+2)}{(2n-1)(2n-3)\dots(2n-m+1)} \quad \text{.....(1) if } m \text{ is even,}$$

and

$$\frac{(2n-m-1)(2n-m-3)\dots(2n-2m+2)}{(2n-1)(2n-3)\dots(2n-m+2)} \quad \text{if } m \text{ is odd.}$$

In the given case, where $m = 6$ and $n = 16$, this is 0·5656 (to 4 places). Hence the probability of at least one match between 2 London teams is 0·4344. The probabilities of 1, 2 or 3 such matches may be calculated separately. The results are 0·3856, 0·0482, and 0·0006.

Followers of the game may recall that one of the matches was Tottenham Hotspur v. Chelsea.

2. Professor Goodstein suggested that the special case where the total number of teams is the square of the number of London teams might have some analogy with the problem of the chance of gaining a prize in a raffle, where there are n^2 tickets and n prizes, if one buys n tickets: it is known that for large values of n, the chance of *not* gaining a prize is approximately e^{-1}.

If we put $m = 2r$ and $n = 2r^2$ in (1) (so that the total number of teams is $4r^2$), we obtain

$$\frac{(4r^2-2r)}{(4r^2-1)} \cdot \frac{(4r^2-2r-2)}{(4r^2-3)} \cdots \frac{(4r^2-4r+2)}{(4r^2-2r+1)}.$$

Now each fraction, expanded in powers of r^{-1}, begins with the terms $1 - \frac{1}{2}r^{-1}$. Hence, as $r \to \infty$, the limiting value of the product is

$$\lim_{r \to \infty} (1 - \tfrac{1}{2}r^{-1})^r = e^{-1/2}.$$

* The draw is made before the replays of drawn matches of the previous round have been played: there were 6 London teams at the time when the draw was made, but one was eliminated in a replay.

† It might be argued that the most general case occurs when the m teams come from an arbitrary geographical area!

It follows that for large values of r the probability of *no* match between 2 London teams is approximately $e^{-1/2}$.

The formula is in fact very nearly correct for quite small values of r. For example, if there are 4 London teams out of 16 (as might happen in the fifth round), the actual probability is 8/13, which is 0·6154, while $e^{-1/2}$ is 0·6065 (both to 4 places).

E. J. F. PRIMROSE

2742. A mean value theorem

The following problem (a simple consequence of Rolle's theorem) appeared in a recent University examination paper:

The function $g(x)$ is continuous in $a < x < b$, and

$$g(a) = 0, \qquad \int_a^b g(t)\,dt = 0. \qquad \text{.....(1)}$$

By considering the function $\varphi(x)$ defined by the relations

$$\varphi(a) = 0, \qquad \varphi(x) = \frac{1}{x-a}\int_a^x g(t)\,dt \qquad (a < x < b),$$

prove that there is at least one ξ in $a < \xi < b$ such that

$$g(\xi) = \frac{1}{\xi - a}\int_a^\xi g(t)\,dt.$$

It is well known that the mean value φ of a function g is in general less irregular in its behaviour than g itself, and the result above merely states the intuitively obvious fact that if the curves $y = g(x)$, $y = \varphi(x)$ start together, and if the former oscillates, then the two will meet at some subsequent point.

We may ask whether the second condition in (1) may be replaced by a simpler condition, for example, by the condition $g(b) = 0$.* The answer is affirmative, and the result in this more general form can be deduced without difficulty from the theorem that a continuous function attains all values between its bounds.

By writing $f(x) = \int_a^x g(t)\,dt$, we obtain an equivalent form of the theorem which might be considered to have independent interest:

If $f(x)$ is differentiable and $f'(x)$ is continuous in $a < x < b$, and $f'(a) = f'(b) = 0$, then there is at least one ξ in $a < \xi < b$ such that

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

This last theorem has a simple geometrical interpretation, namely that if the curve $y = f(x)$ has a continuously turning tangent in $a < x < b$, and if the tangents at $x = a$ and $x = b$ are parallel, then there is an intermediate point ξ such that the tangent there passes through the point a. It is easily seen that the condition that the tangents at a, b be parallel is the natural one here, and in this form the theorem is even more obvious on intuitive grounds than is the integral form. There is, however, one last extension which we can make—we can omit the hypothesis that f' is continuous. The resulting theorem is a good deal less obvious, and the proof is perhaps more difficult. We have in fact the

* The second condition in (1) implies that g must vanish somewhere between a and b.

THEOREM. *If $f(x)$ is differentiable in $a < x < b$, and $f'(a) = f'(b)$, then there exists a point ξ in $a < \xi < b$ such that the tangent to the curve $y = f(x)$ at the point ξ passes through the point a, i.e. such that*

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

We may suppose that $f'(a) = f'(b) = 0$, for if this is not the case we work with $f(x) - xf'(a)$. Consider now the function ψ defined by the relations

$$\psi(a) = f'(a) = 0, \qquad \psi(x) = \frac{f(x) - f(a)}{x - a} \qquad (a < x \leqslant b).$$

Evidently ψ is continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x \leqslant b$, and

$$\psi'(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f'(x)}{x - a} \qquad (a < x \leqslant b).$$

It is therefore sufficient to prove that there is some point ξ in $a < \xi < b$ such that $\psi'(\xi) = 0$.

This is an immediate consequence of Rolle's theorem if $\psi(b) = 0$. Suppose then that $\psi(b) > 0$, so that $\psi'(b) = -\psi(b)/(b - a) < 0$. Then there exists x_1 in $a < x_1 < b$ such that $\psi(x_1) > \psi(b)$. Since ψ is continuous in $a \leqslant x \leqslant x_1$ and $\psi(a) < \psi(b) < \psi(x_1)$, there is a point x_2 in $a < x_2 < x_1$ such that $\psi(x_2) = \psi(b)$, and the required result now follows from Rolle's theorem applied to the function ψ in the interval (x_2, b). A similar argument applies if $\psi(b) < 0$, and this completes the proof.

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2743. An inequality

In connexion with Mr C. V. Durell's letter (*Math. Gaz.* 50 (1956), 266), some further information about the alleged inequality

$$f_n(x) \equiv \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \ldots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} > \tfrac{1}{2}n, \quad \ldots\ldots(1)$$

where $x_i > 0$ for $i = 1, 2, \ldots, n$, can be found in the *American Mathematical Monthly* 63 (1956), 191–192. Here Lighthill's example of the falsity of (1) for $n = 20$ is given by F. H. Northover, and it is stated that (1) has been proved to be true for $n = 5$ by C. R. Phelps.

Let $\mu(n)$ be the greatest lower bound of $f_n(x)$ for $x_i > 0$ ($i = 1, 2, \ldots, n$), so that $\mu(n) = \frac{1}{2}n$ for $n < 5$.

By taking n even and choosing

$$x_{2r-1} = 1 + \varepsilon a_r, \qquad x_{2r} = \varepsilon a_{r-1} \qquad (1 \leqslant r \leqslant m = \tfrac{1}{2}n),$$

with positive ε, a_r and $a_0 = a_m$, Professor Lighthill has shown* that, more generally,

$$\mu(n) < \tfrac{1}{2}n \qquad (n > 12, \, n \text{ even}). \qquad \ldots\ldots(2)$$

In the particular case $n = 16$, I have made a rough numerical calculation based on his method and find that

$$\mu(16) < 16c, \qquad \text{where } c = \tfrac{1}{2} - 7 \times 10^{-6}. \qquad \ldots\ldots(3)$$

The following additional results may be of interest.

* Private communication.

By taking

$$x_{kn+r} = x_r \qquad (0 < k < N, \quad 0 < r < n),$$

we see that $\mu(Nn) < N\mu(n)$, so that

$$\frac{\mu(Nn)}{Nn} < \frac{\mu(n)}{n}. \qquad \text{.....(4)}$$

Also, if $x_i < x_n$ $(i < n)$ and $r > 1$, then

$$f_{n+r}(x_1, x_2, \ldots, x_{n-1}, x_n, x_n, \ldots, x_n) - f_n(x_1, x_2, \ldots, x_n)$$

$$= \frac{x_{n-1}}{2x_n} + \tfrac{1}{2}(r-1) + \frac{x_n}{x_n + x_1} - \frac{x_{n-1}}{x_n + x_1}$$

$$= \frac{x_{n-1}(x_1 - x_n)}{2x_n(x_n + x_1)} + \frac{x_n}{x_n + x_1} + \tfrac{1}{2}(r-1)$$

$$< 0 + 1 + \tfrac{1}{2}(r-1) = \tfrac{1}{2}(r+1),$$

so that

$$\mu(n+r) < \mu(n) + \tfrac{1}{2}(r+1). \qquad \text{.....(5)}$$

We deduce from (3), (4) and (5), that

$$\mu(16m + r) < \mu(16m) + \tfrac{1}{2}(r+1) < 16mc + \tfrac{1}{2}(r+1)$$
$$< \tfrac{1}{2}(16m + r)$$

if $m > X = \{16(1 - 2c)\}^{-1}$. Hence

$$\mu(n) < \tfrac{1}{2}n \qquad \text{for all } n > 16X.$$

In particular, (1) is false also for sufficiently large odd n.

Further, if $m = qn + r$, where $q > 1$, $0 < r < n$, then, by (4) and (5),

$$\frac{\mu(m)}{m} < \frac{\mu(qn)}{qn} \cdot \frac{qn}{m} + \frac{r+1}{2m} < \frac{\mu(n)}{n} + \frac{n}{2m},$$

so that

$$\limsup_{m \to \infty} \frac{\mu(m)}{m} < \frac{\mu(n)}{n},$$

and hence

$$\limsup_{m \to \infty} \frac{\mu(m)}{m} < \inf_{n \geqslant 2} \frac{\mu(n)}{n} < \liminf_{n \to \infty} \frac{\mu(n)}{n}.$$

From this we deduce that $\mu(n)/n$ tends to a limit as $n \to \infty$, and that

$$\lim_{n \to \infty} \frac{\mu(n)}{n} = \inf_{n \geqslant 2} \frac{\mu(n)}{n} < c < \tfrac{1}{2}.$$

By somewhat complicated methods involving convex functions, I have been able to obtain a result in the opposite direction, namely

$$\mu(n) > \tfrac{1}{2}(2\sqrt{2} - 1)n \fallingdotseq 0{\cdot}304n,$$

for all $n > 1$. This is certainly not best possible.

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2744. A curious triangle

The following problem was suggested by the late Prof. A. A. K. Ayyangar (*Mathematical Problem Papers*, 1923):

If a triangle be similar to the triangle formed by its medians, prove that the sum of the squares on two sides of the triangle is equal to twice the square on the third side.

The proof is simple. If $a > b > c$ and the medians AD, BE, CF are of lengths l, m, n, then $n > m > l$ and Apollonius' Theorem gives

$$4l^2 = 2b^2 + 2c^2 - a^2 = \lambda c^2$$
$$4m^2 = 2c^2 + 2a^2 - b^2 = \lambda b^2$$
$$4n^2 = 2a^2 + 2b^2 - c^2 = \lambda a^2$$

from which it follows that $\lambda = 3$ and $a^2 + c^2 = 2b^2$.

Prof. Ayyangar gives the following additional results, all easily proved:

(1) If G is the centroid, $BDGF$ is a cyclic quadrilateral.

(2) Cot A, cot B, cot C are in Arithmetic Progression.

(3) The triangle formed by the circumcentres of triangles AGB, AGC, BGC is similar to the triangle ABC.

The corresponding result for nedians (M.G. XL. 109) appears to be

$$na^2 = (n-1)c^2 + b^2.$$

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2745. A note on factorization

Eratosthenes applied the method of "sifting" the natural numbers by successively removing the multiples of 2, 3, etc., and much has since been written on the distribution of primes. but no simple law of the sequence of prime numbers (or of the sequence of composite numbers) has as yet been found. The natural way of testing the prime character of any number is that of trial and error, using all primes smaller than the square root of the given number. A simpler method, used by Fermat in which no division is necessary, is given in the present note.

Of course we only need to consider *odd* positive numbers n. Every n may be written in the form

$$n = \alpha\beta \qquad \text{.....(1)}$$

in which α and β are odd numbers satisfying the inequalities

$$\left.\begin{aligned} 1 < \alpha < \sqrt{n}, \\ \sqrt{n} < \beta < n. \end{aligned}\right\} \qquad \text{.....(2)}$$

Equation (1) may be written as

$$\begin{aligned} n &= \alpha\beta \\ &= \left(\frac{\beta+\alpha}{2}\right)^2 - \left(\frac{\beta-\alpha}{2}\right)^2 \\ &= b^2 - a^2 \end{aligned} \qquad \text{.....(3)}$$

with the positive integers a and b defined by

$$\left.\begin{aligned} a &= \frac{\beta-\alpha}{2}, & b &= \frac{\beta+\alpha}{2}, \\ \alpha &= b - a, & \beta &= b + a. \end{aligned}\right\} \qquad \text{.....(4)}$$

This means that there is a unique correspondence between factorizations (1) of n, and representations of the form (3). From (2) follows

$$a < \frac{n-1}{2}, \qquad \text{.....(5)}$$

and also
$$b < \frac{n+\sqrt{n}}{2}; \qquad \text{.....(6)}$$

but this latter inequality can be improved on.

For this purpose we keep n fixed, and consider α for the moment as a continuous variable in the formulae (1) to (4) where it occurs. Then β, a and b also vary continuously and decrease monotonically with increasing α in the range defined by (2).
Since

$$2b = \beta + \alpha = \frac{n}{\alpha} + \alpha,$$

the inequality (6) can therefore be refined to

$$b \leq \frac{1}{2}\left(\frac{n}{\alpha_{\min}} + \alpha_{\min}\right) = \frac{n+1}{2} \qquad \text{.....(7)}$$

where $\alpha_{\min}$ denotes the lower bound of α in the given range. If n is prime, $\alpha = 1$ and $\beta = n$, and the equalities in (5) and (7) hold. (It may be noted in passing that in this case $b - a = 1$, whereas for composite n a representation of the form (3) exists with $b - a > 1$. Using this property as a basis for determining the character of the natural numbers could of course lead to a scheme similar to that of Eratosthenes.)

We now make use of the identity

$$a^2 = \sum_{r=1}^{a} (2r-1) \qquad \text{.....(8)}$$

which holds for positive integral values of a. On substitution in (3), this gives

$$b^2 = n + \sum_{r=1}^{a} (2r-1). \qquad \text{.....(9)}$$

If n is composite, i.e. if there exists a factorization (1) with $\alpha > 1$, then

$$\left.\begin{aligned} a &< \frac{n-1}{2}, \\ b &< \frac{n+1}{2}. \end{aligned}\right\} \qquad \text{.....(10)}$$

This means for an a satisfying (10), the right-hand side of (9) equals the square of an integer $\left(\text{which, incidentally, is smaller than } \frac{n+1}{2}\right)$, resulting in the following rule:

If to n are added successively the odd positive numbers up to the ath, where a satisfies (10), *and the sum is at any stage the square of an integer, then n is composite; otherwise it is prime.*

The sole requirement for the application of this process is, therefore, a table of squares.

Since it is a simple matter to test n for the presence of small factors (e.g. 3 and 5), it may be convenient to apply the above process only to detect larger factors. This means that $\alpha_{\min}$ in (7) and in the corresponding formula for a may be replaced by a larger value. Thus

$$a < \frac{1}{2}\left(\frac{n}{\alpha_{\min}} - \alpha_{\min}\right),$$

$$b < \frac{1}{2}\left(\frac{n}{\alpha_{\min}} + \alpha_{\min}\right),$$

where $\alpha_{\min}$ may be chosen to equal 3, 5, 7, etc., and (10) becomes

$$a < \frac{1}{2}\left(\frac{n}{\alpha_{\min}} - \alpha_{\min}\right).$$

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2746. Randomness

The difficulty of interpreting the term "at random" is well illustrated in the problem of finding the mean length of a large number of chords drawn at random in a circle of unit radius.

In this case "at random" may be interpreted in a number of ways. Three examples are given below. Each of these might be intellectually satisfying if the others did not exist.

1. Consider all chords perpendicular to a given diameter. Then since the chords are drawn at random we may assume that they cut the diameter with uniform density and the mean chord length is:

$$2\int_0^1 \sqrt{1-x^2}\,dx = \tfrac{1}{2}\pi = 1{\cdot}57$$

2. Consider all chords drawn through a given point on the circumference. We may assume that they are uniformly distributed in angles to the diameter. This is equivalent to assuming that the other ends of the chords are uniformly distributed about the circumference. In this case, the mean chord length is:

$$\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \cos\theta\,d\theta = \frac{4}{\pi} = 1{\cdot}27$$

3. Apart from the centre, every point in a circle is the mid-point of a unique chord. There is, therefore, a one-one correspondence between the points and chords in a circle and selecting chords at random is equivalent to selecting chord centres at random. If we assume that random chord centres are uniformly distributed in the circle, the mean chord length is:

$$4\int_0^1 x\sqrt{1-x^2}\,dx = \frac{4}{3} = 1{\cdot}33$$

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2747. On Notes 2023, 2094, 2299

Three proofs of the identity

$$\frac{c_1}{1} - \frac{c_2}{2} + \ldots + (-1)^{n-1}\frac{c_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

have been given in the above Notes; the first, 2023, contains an elegant proof by Mr. Parameswaran, using Partial Fractions; the second, 2094, a proof by Dr. Busbridge, using Integration; and the third, 2299, a generalization by Mr. Swinden. The following direct proof differs from all these, and is, I think, simpler and more elementary than any.

We know that $${}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1}. \qquad \ldots\ldots(1)$$

Dividing by r, we obtain $\frac{{}_nC_r}{r} = \frac{{}_{n-1}C_r}{r} + \frac{{}_nC_r}{n}$, ...(2)

since $${}_{n-1}C_{r-1}/r = {}_nC_r/n.$$

Now if $s_n = \frac{c_1}{1} - \frac{c_2}{2} + \ldots + (-1)^{n-1}\frac{c_n}{n}$, putting $r = 1, 2, \ldots n$ in (2) and adding and subtracting alternately, we obtain

$$s_n - s_{n-1} = \frac{1}{n}\{{}_nC_1 - {}_nC_2 + \ldots + (-)^{n-1}{}_nC_n\}$$
$$= \frac{1}{n} \qquad \ldots\ldots(3)$$

From this, and $s_0 = 0$, we at once derive the required result

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

Two further curious results can be obtained by the same method.
(*a*) Divide (1) by r^2, instead of r, and proceed as before.
We obtain

$$\frac{c_1}{1^2} - \frac{c_2}{2^2} + \frac{c_3}{3^2} - \ldots + (-1)^{n-1}\frac{c_n}{n^2} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \ldots$$
$$+ \frac{1}{n}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$

This may also be proved by Dr. Busbridge's method (No. 3 of Note 2094), using the integral

$$\int_0^1 \frac{dy}{y}\int_0^y \frac{1-(1-x)^n}{x}\,dx = \int_0^1 \frac{dy}{y}\int_{1-y}^1 \frac{1-t^n}{1-t}\,dt.$$

(b) If $t_n = \frac{c_1}{1} + \frac{c_2}{2} + \frac{c_3}{3} + \ldots + \frac{c_n}{n}$, we obtain from (2) with $r = 1, 2 \ldots n$ and repeated addition

$$t_n - t_{n-1} = \frac{1}{n}\{c_1 + c_2 + \ldots + c_n\} = \frac{1}{n}(2^n - 1). \qquad \ldots\ldots(4)$$

From (4), and $t_0 = 0$, we obtain

$$t_n = \frac{c_1}{1} + \frac{c_2}{2} + \frac{c_3}{3} + \ldots + \frac{c_n}{n} = 1 + \frac{1}{2}(2^2 - 1) + \ldots + \frac{1}{n}(2^n - 1).$$

This last result also follows from the equation

$$\int_1^2 \frac{x^n - 1}{x - 1}\,dx = \int_0^1 \frac{(1+t)^n - 1}{t}\,dt.$$

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2748. A problem on cubic equations*

There can be no doubt that some among the Tripos candidates of Forsyth's year would have been ashamed to plunge blindly into the childish algebra which Mr. Wilson handsomely allows to be "easily within their compass"; they would have taken their bearings before plunging. The problem is:

* See M. G. XXXIX, p. 280, and XL, p. 293.

Given the equations

$$x^3 - 3p_1x^2 + 3q_1x - r_1 = 0, \qquad x^3 - 3p_2x^2 + 3q_2x - r_2 = 0, r \quad \ldots\ldots(1)$$

whose roots are $\alpha_1, \beta_1, \gamma_1$ *and* $\alpha_2, \beta_2, \gamma_2$, *find the equations whose roots* λ', μ', ν' *and* λ'', μ'', ν'' *are*

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \qquad \alpha_1\beta_2 + \beta_1\gamma_2 + \gamma_1\alpha_2, \qquad \alpha_1\gamma_2 + \beta_1\alpha_2 + \gamma_1\beta_2$$

and

$$\alpha_1\alpha_2 + \beta_1\gamma_2 + \gamma_1\beta_2, \qquad \alpha_1\gamma_2 + \beta_1\beta_2 + \gamma_1\alpha_2, \qquad \alpha_1\beta_2 + \beta_1\alpha_2 + \gamma_1\gamma_2.$$

If the required equations are $\varphi' = 0, \varphi'' = 0$, the function φ' is transformed into φ'' by the interchange of any two of the roots $\alpha_1, \beta_1, \gamma_1$ or of any two of the roots $\alpha_2, \beta_2, \gamma_2$. Hence $\varphi' - \varphi''$ has each of the six differences $\beta_1 - \gamma_1, \gamma_1 - \alpha_1, \alpha_1 - \beta_1, \beta_2 - \gamma_2, \gamma_2 - \alpha_2, \alpha_2 - \beta_2$ for a factor, and since only the constant terms in φ' and φ'' are of degree as high as six in the original roots, $\varphi' - \varphi''$ is a numerical multiple of the product of the six differences. That is, if

$$\bar{\omega}_1 = (\beta_1 - \gamma_1)(\gamma_1 - \alpha_1)(\alpha_1 - \beta_1), \qquad \bar{\omega}_2 = (\beta_2 - \gamma_2)(\gamma_2 - \alpha_2)(\alpha_2 - \beta_2),$$

then

$$\lambda'\mu'\nu' - \lambda''\mu''\nu'' = k\bar{\omega}_1\bar{\omega}_2,$$

where k is a numerical constant, and the required equations have the form

$$x^3 - \mathrm{A}x^2 + \mathrm{M}x - \mathrm{N} \mp \tfrac{1}{2}k\bar{\omega}_1\bar{\omega}_2 = 0$$

where A, M, N are symmetric both in $\alpha_1, \beta_1, \gamma_1$ and in $\alpha_2, \beta_2, \gamma_2$.

To subtract p_1 from each of $\alpha_1, \beta_1, \gamma_1$ is to subtract $3p_1p_2$ from each of λ', μ', ν' and from each of λ'', μ'', ν''. It follows that if we solve our problem for the pair of equations

$$x^3 + 3u_1x - v_1 = 0, \qquad x^3 + 3u_2x - v_2 = 0 \quad \ldots\ldots(2)$$

we can write down the solution for the pair (1). Half the algebra disappears at a stroke, and little is left of the other half when we notice that for the equation

$$x^3 + 3ux - v = 0 \quad \ldots\ldots(3)$$

there is no non-zero symmetric function of the first degree in the roots, every symmetric function of degree two is a multiple of u, and every symmetric function of degree three is a multiple of v. We infer that for the pair of equations (2) the required equations have the form

$$x^3 + au_1u_2x - bv_1v_2 \mp \tfrac{1}{2}k\bar{\omega}_1\bar{\omega}_2 = 0.$$

To complete the solution, let us suppose each of the equations (2) to be the equation (3), with roots α, β, γ. Then $\lambda' = -6u, \mu' = \nu' = 3u$, and the equation with these roots is

$$x^3 - 27u^2x + 54u^3 = 0,$$

to be identified with

$$x^3 + au^2x - bv^2 - \tfrac{1}{2}k\bar{\omega}^2 = 0.$$

Thus $a = -27$, and if we assume, as we should in an examination, the formula $\bar{\omega}^2 = -27(4u^3 + v^2)$, we have immediately $k = 1, b = 27/2$. We can avoid the assumption by using also λ'', μ'', ν'', which are $\alpha^2 + 2\beta\gamma, \beta^2 + 2\gamma\alpha, \gamma^2 + 2\alpha\beta$; the equation with these roots is

$$27v^3 + 9uv(x + 3u)^2 - v(x + 3u)^3 = 0,$$

that is,

$$x^3 - 27u^2x - 54u^3 - 27v^2 = 0,$$

to be identified with

$$x^3 - 27u^2x - bv^2 + \tfrac{1}{2}k\bar{\omega}^2 = 0.$$

Thus $2b = 27$, and $k\bar{\omega}^2 = -27\Delta$, where $\Delta = 4u^3 + v^2$. The constant k is not determined, but it is not needed, for the term we have to evaluate is $k\bar{\omega}_1\bar{\omega}_2$, and

$$k^2\bar{\omega}_1{}^2\bar{\omega}_2{}^2 = k\bar{\omega}_1{}^2 \cdot k\bar{\omega}_2{}^2 = 27^2\Delta_1\Delta_2.$$

For the pair of equations (2), the required equations are

$$2x^3 - 54u_1u_2x - 27\{v_1v_2 \pm \sqrt{(\Delta_1\Delta_2)}\} = 0,$$

and therefore for the original pair of equations (1), the required equations are

$$2(x - 3p_1p_2)^3 - 54u_1u_2(x - 3p_1p_2) - 27\{v_1v_2 \pm \sqrt{(\Delta_1\Delta_2)}\} = 0,$$

where, for each suffix,

$$u = q - p^2, \qquad v = r - 3pq + 2p^3, \qquad \Delta = 4u^3 + v^2.$$

To find the value of k and so to prove that $\bar{\omega}^2 = -27\Delta$ we need only consider the case in which $\alpha = 0$ in the repeated equation (3); in this case, $\lambda'\mu'\nu' = \beta^2\gamma^2(\beta^2 + \gamma^2)$, $\lambda''\mu''\nu'' = 2\beta^3\gamma^3$, and $\lambda'\mu'\nu' - \lambda''\mu''\nu'' = \beta^2\gamma^2(\beta - \gamma)^2 = \bar{\omega}^2$ immediately.

The author of Note 2654 (XL, p. 293) asserts that it is a much more difficult problem to find the equation whose roots are *the same combinations* of the squares of α_1, β_1, γ_1 and α_2, β_2, γ_2 and he devotes some forty lines of algebra to establishing, possibly to his own surprise, but not at all to mine, that the equations whose roots are the same combinations of the nth powers of α_1, β_1, γ_1 and α_2, β_2, γ_2 are found by writing, for each suffix, $\Sigma\alpha^n$, $\Sigma\alpha^n\beta^n$ and $\alpha^n\beta^n\gamma^n$ for p, q, r in the above equations. To find the equation $x^3 - p_nx^2 + q_nx - r_n = 0$ whose roots are the nth powers of the roots α, β, γ of the equation $x^3 - px^2 + qx - r = 0$, we note that $\alpha^3 = p\alpha^2 - q\alpha + r$ and therefore $p_n = pp_{n-1} - qp_{n-2} + rp_{n-3}$ with $p_0 = 3$, $p_1 = p$, $p_2 = p^2 - 2q$; and since the equation with roots r/α, r/β, r/γ is $x^3 - qx^2 + prx - r^2 = 0$ therefore *by the same recursion* $q_n = qq_{n-1} - prq_{n-2} + r^2q_{n-3}$ with $q_0 = 3$, $q_1 = q$, $q_2 = q^2 - 2pr$.

E. H. NEVILLE

2749. MXCCLI

The above uncommon sequence of letters on the title-page of the anonymous first edition of Rowe's *Introduction to the Doctrine of Fluxions* fails to give the date of publication. Fortunately, Mr. D. E. Rhodes tells me, the preface is dated precisely, and shows that the obvious correction is the one to make; in the British Museum Catalogue, in Halkett & Laing, and in Sotheran's *Bibliotheca Chemico-Mathematica*, to name only three authorities, the year is entered as 1751.

Up to this point we have only an example of human fallibility, and an instance of the famous Theorem of Dupin—Auguste, not Charles—that objects most prominent are the most likely to be overlooked. But there is a sequel. In Cajori's monograph, *A History of the Conceptions of ... Fluxions in Great Britain ...*, 1919, a transcription of the title of this *Introduction* ends with a date written as M.DCC.XLI, with no hint that this date is a conjectural emendation which differs from the accepted one.

The adoption of the earlier date, coupled with a convenient practice of dealing with various editions of one book in the same section, has the immediate effect of bringing Cajori's discussion not only of the *Introduction* in its original form but also of the second edition (1757) and the third (1767) into his chapter on "the books which preceded the publication of Maclaurin's *Treatise of Fluxions*, 1742" instead of into his chapter on mid-century text-books.

Since in Cajori's view Maclaurin, like Hardy 50 years ago, set an entirely new standard by which expository writings could properly be assessed, Cajori on his own principles judges Rowe far too leniently.

Does it matter that John Rowe should be overrated in the brief moments in which he escapes oblivion? Perhaps not, but Cajori's reputation is involved no less than Rowe's. Either Cajori himself blundered, or he took inadequate precautions against unreliable help, a grave fault in a historian, and one which provoked Greenstreet a few years later to offer him (*Gazette*, XIV, p. 251) "condolences on the way in which his proof-reader has neglected his business." Whose business? An ambiguity as wicked as this is not accidental.

E. H. NEVILLE

2750. Differential equations with constant coefficients

(1) A method for finding a particular integral of the linear differential equation

$$f(D)y = x^m \qquad \text{where} \quad F(D) = a_0D^k + a_1D^{k-1} + \ldots + a_k$$

The method is particularly useful in cases where $F(D)$ does not factorize.

Differentiate the equation m times, obtaining finally

$$(a_0D^{k+m} + \ldots + a_kD^m)y = m!$$

We now obtain a particular integral of this set of $m + 1$ equations.

A particular integral of the last equation will be given by a particular solution of $a_kD^my = m!$

Hence we can substitute $m!/a_k$ for D^my and zero for higher derivatives in the preceding equation.

We now obtain $D^{m-1}y$ as a function of x. Substitute this expression and its derivatives for $D^{m-1}y$; D^my etc. in the preceding equation and obtain $D^{m-2}y$ as a function of x.

Substitute in preceding equation as before. After m steps substituting in the original equation we arrive at a particular integral of the original equation.

Example Solve $(D^2 + 3D + 1)y = x^2$ .....(1)

Differentiating: $(D^3 + 3D^2 + D)y = 2x$ .....(2)

$(D^4 + 3D^3 + D^2)y = 2$ .....(3)

Obtain simplest integral of (3) from $D^2y = 2$

Substituting in (2) $0 + 6 + Dy = 2x$

$Dy = 2x - 6$

Substituting in (1) $2 + 6x - 18 + y = x^2$

$y = x^2 - 6x + 16$

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2751. On impulse theorems

In Note 2272, Weston and Whitfield have discussed Robin's theorem from a geometrical viewpoint; in the present note some other theorems on impulses are treated in a similar manner.

In this geometrical treatment, an initial motion, with kinetic energy T, of an unconstrained dynamical system of n degrees of freedom is represented by a vector $\overrightarrow{OP}$, of length $\sqrt{T}$, in n-dimensional Euclidean space; similarly the vectors $\overrightarrow{OP'}$ (length $\sqrt{T'}$), OP'' (length $\sqrt{T''}$) respectively represent the Bertrand and Kelvin comparison motions of the constrained system. Then the theorems of Bertrand, Kelvin and Sir Geoffrey Taylor (Ramsey, *Dynamics*, Vol. II, §§ 9.43, 9.44) respectively imply that the difference-motion vectors $\overrightarrow{P'P}$, $\overrightarrow{PP''}$, $\overrightarrow{P'P''}$ have lengths $\sqrt{(T-T')}$, $\sqrt{(T''-T)}$, $\sqrt{(T''-2T+T')}$ respectively, or that the angles $OP'P$, OPP'', $PP'P''$ are right angles.

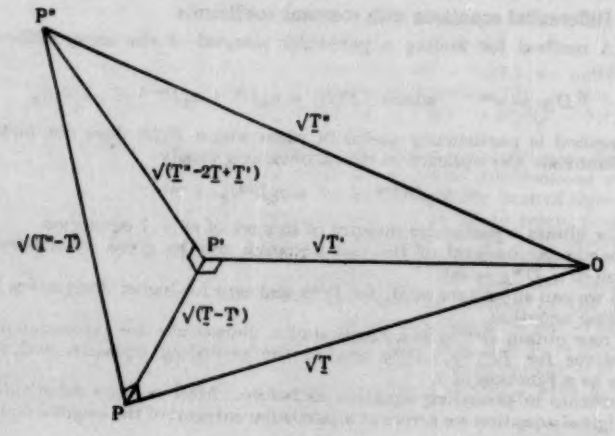

FIG. 1

These conditions alone, however, are insufficient to determine a possible figure $OPP'P''$; geometrical considerations show that an extra condition necessary and sufficient for this purpose is:

$$OP''-OP' < P'P'',$$

which is equivalent to:

$$T < \sqrt{(T'T'')}; \qquad \text{.....(1)}$$

that is, the kinetic energy of an unconstrained motion is less than, or equal to, the geometric mean of the kinetic energies of the Bertrand and Kelvin comparison motions arising from the imposition of the same constraints. Equality occurs whenever O, P', P'' are collinear, that is, whenever only one effective degree of freedom remains after imposition of the constraints (for a fully-worked example, see Milne, *Vectorial Mechanics*, § 440).

Independent proofs of (1) may be based on the equations

$$\tfrac{1}{2}\sum_{r=1}^{n} p_r''\dot{q}_r' = \tfrac{1}{2}\sum_{r=1}^{n} p_r'\dot{q}_r'' = T, \qquad \text{.....(2)}$$

where single primes refer to the Bertrand motion and double primes to the Kelvin motion (the first equation of (2) is the well-known "reciprocal relation"). The second equation of (2) is proved by observing that, from the conditions of the theorems of Bertrand and Kelvin, $p_r' = p_r$, $\dot{q}_r'' = \dot{q}_r$ for the unconstrained co-ordinates, and $p_r = 0$, $\dot{q}_r'' = 0$ for the constrained co-ordinates, where p_r, $\dot{q}_r$ belong to the original motion; hence $p_r'\dot{q}_r'' = p_r\dot{q}_r$ for all co-ordinates and

$$\tfrac{1}{2}\sum_{r=1}^{n} p_r'\dot{q}_r'' = \tfrac{1}{2}\sum_{r=1}^{n} p_r\dot{q}_r = T.$$

The inequality (1) follows immediately (as pointed out by the referee) from

$$\overrightarrow{OP'}.\overrightarrow{OP''} < |\overrightarrow{OP'}|\,|\overrightarrow{OP''}| = \sqrt{(T'T'')};$$

for the scalar product (whose symmetry implies the "reciprocal relation") is equal to the members of (2) because $p_r/\sqrt{2}$ and $\dot{q}_r/\sqrt{2}$ are covariant and contravariant vector components (Note 2272). Alternatively, the proof of (1) may be completed, independently of geometrical considerations, by using (2) to show that the kinetic energy of a linear combination of α' times the Bertrand motion with α'' times the Kelvin motion is given by

$$\tfrac{1}{2}\sum_{r=1}^{n}\{(\alpha' p_r' + \alpha'' p_r'')(\alpha'\dot{q}_r' + \alpha''\dot{q}_r'')\} = T'\alpha'^2 + T''\alpha''^2 + 2T\alpha'\alpha'',$$

and then observing that this must be positive or zero for all real α', α''.

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2752. Approximate integration

The formulae quoted in Note 2489 are members of families of such formulae, as may readily be shown by developing a method for which my father, C. H. Wickens, was responsible.

Let $$f(x) = p_0 + p_1x + p_2x^2 + p_3x^3$$

Then $$\int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(x)\,dx = 5p_0 + \frac{125}{12}p_2 \qquad (1)$$

We wish to express the value of this integral in the form

$$z_{-2\frac{1}{2}}.f(-2\tfrac{1}{2}) + z_{-1\frac{1}{2}}.f(-1\tfrac{1}{2}) + z_{-\frac{1}{2}}.f(-\tfrac{1}{2}) + z_{\frac{1}{2}}.f(\tfrac{1}{2}) + z_{1\frac{1}{2}}.f(1\tfrac{1}{2}) + z_{2\frac{1}{2}}.f(2\tfrac{1}{2}) \qquad (2)$$

where the z's represent weights to be determined. The values of $f(-2\frac{1}{2})$, etc. could be written down in terms of the p's, these values substituted in (2), and the coefficients of the p's equated. The following method, however, greatly reduces the arithmetic involved.

$$\int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(x)\,dx = \int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(-x)\,dx$$

Hence from (2) we have

$$\sum_{1}^{3}(z_{m-\frac{1}{2}} - z_{-m+\frac{1}{2}})[f(m-\tfrac{1}{2}) - f(-m+\tfrac{1}{2})] = 0$$

and, equating the coefficients of p_1 and of p_3, (those of p_0 and p_2 are zero),

$$(z_{\frac{1}{2}} - z_{-\frac{1}{2}}) + 3(z_{1\frac{1}{2}} - z_{-1\frac{1}{2}}) + 5(z_{2\frac{1}{2}} - z_{-2\frac{1}{2}}) = 0 \qquad (3)$$

$$(z_{\frac{1}{2}} - z_{-\frac{1}{2}}) + 27(z_{1\frac{1}{2}} - z_{-1\frac{1}{2}}) + 125(z_{2\frac{1}{2}} - z_{-2\frac{1}{2}}) = 0 \qquad (4)$$

Further, $$\int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(x)\,dx = \tfrac{1}{2}\left[\int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(x)\,dx + \int_{-2\frac{1}{2}}^{2\frac{1}{2}} f(-x)\,dx\right]$$

and therefore $$5p_0 + \frac{125}{12}p_2 = \tfrac{1}{2}\sum_{1}^{3}(z_{m-\frac{1}{2}} + z_{-m+\frac{1}{2}})[f(m-\tfrac{1}{2}) + f(-m+\tfrac{1}{2})]$$

and, equating the coefficients of p_0 and p_2, (those of p_1 and p_3 are zero),

$$(z_{\frac{1}{2}} + z_{-\frac{1}{2}}) + (z_{1\frac{1}{2}} + z_{-1\frac{1}{2}}) + (z_{2\frac{1}{2}} + z_{-2\frac{1}{2}}) = 5 \qquad (5)$$

$$(z_{\frac{1}{2}} + z_{-\frac{1}{2}}) + 9(z_{1\frac{1}{2}} + z_{-1\frac{1}{2}}) + 25(z_{2\frac{1}{2}} + z_{-2\frac{1}{2}}) = \frac{125}{3} \qquad (6)$$

From equations (3) and (4) it follows that, if $z_{\frac{1}{2}} = z_{-\frac{1}{2}}$, then $z_{1\frac{1}{2}} = z_{-1\frac{1}{2}}$ and $z_{2\frac{1}{2}} = z_{-2\frac{1}{2}}$. The further condition that the weights should be symmetrical may accordingly be imposed and equations (5) and (6) may then be reduced to

$$z_{1\frac{1}{2}} = \frac{55}{24} - 3z_{2\frac{1}{2}}, \quad z_{\frac{1}{2}} = \frac{5}{24} + 2z_{2\frac{1}{2}}$$

We have thus established that if y is a function with third order differences constant

$$\int_a^{a+5w} y\,dx = w\sum_{a}^{a+5w} z_x \cdot y_x$$

where

$$z_a = z_{a+5w} = z$$

$$z_{a+w} = z_{a+4w} = \frac{55}{24} - 3z$$

$$z_{a+2w} = z_{a+3w} = \frac{5}{24} + 2z$$

and z can have any value.

In practice, it is desirable that z_k should be positive for all values of k.

The expression given in Note 2489 is obtained by giving z the value 5/16. Simpler expressions are however obtainable if z be given the values 0, 5/24, 5/12 or 5/8, and with these values of z those of $z_a, \ldots, z_{a+5w}$ can be written as

$$\begin{array}{llllll} 5/24\,\{0, & 11, & 1, & 1, & 11, & 0\} \\ 5/24\,\{1, & 8, & 3, & 3, & 8, & 1\} \\ 5/24\,\{2, & 5, & 5, & 5, & 5, & 2\} \\ 5/24\,\{3, & 2, & 7, & 7, & 2, & 3\} \end{array}$$

The addition of any multiple of $\{1, -3, 2, 2, -3, 1\}$ to the weights will not affect the value of the integral.

The definite integral of a function with fourth order differences constant may, with appropriate weights, be expressed in the same form, but the weights will be symmetrical only for the particular set of weights 5/288 {19, 75, 50, 50, 75, 19}. The method employed above may be used to express the definite integral of any function with nth order differences constant in terms of $(n + 2)$ equidistant values of the function; it will be found that in such cases the weights of each set are symmetrical for n odd but only one set of weights is symmetrical for n even.

It may likewise be shown that if y is a function with third order differences constant

$$\int_a^{a+7w} y\,dx = w \sum_a^{a+7w} z_k \cdot y_k$$

where

$$\begin{aligned} z_a &= z_{a+7w} = z \\ z_{a+w} &= z_{a+6w} = (p/2 + 77/48) + (q/2 - 5/2)z \\ z_{a+2w} &= z_{a+5w} = (91/48 - 3p/2) + (3/2 - 3q/2)z \\ z_{a+3w} &= z_{a+4w} = p + q \cdot z \end{aligned}$$

and z, p or q can have any values.

If $z = 7/24$, $p = 7/8$, $q = 1$ we obtain the values given in Note 2489 namely

$$7/24\ \{1,\ 5,\ 2,\ 4,\ 4,\ 2,\ 5,\ 1\}$$

and we can also obtain the following values of equal simplicity

$$7/24\ \{2,\ 2,\ 5,\ 3,\ 3,\ 5,\ 2,\ 2\}$$

while the addition of any multiple of

$$\{1,\ -3,\ 3,\ -1,\ -1,\ 3,\ -3,\ 1\}$$

to the weights will not affect the value of the integral.

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2753. Powers of $\sum_1^n r$

Two recent notes (2678 and 2687) and the article by Miss Edmonds (October 1957, pp. 187–188) have given a new look to the familiar identity

$$S_3 = S_1^2 \quad \text{where} \quad S_k = \sum_1^n r^k.$$

Another surprising identity Miss Edmonds mentions is $S_5 + S_7 = 2S_1^4$ and it is natural to inquire whether these are isolated results. An expansion for the pth power of S_k of the form

$$S_k{}^p = a_1 S_{(k+1)p-1} + a_2 S_{(k+1)p-3} + \cdots$$

can be derived with the use of Bernoulli polynomials and this includes the given identities when $k = 1$ and $p = 2$, 4. However the formula for $S_1{}^p$ may be found by elementary means. Thus for integral p

$$\tfrac{1}{2}(m+1)^p - \tfrac{1}{2}(m-1)^p = {}^pC_1 m^{p-1} + {}^pC_3 m^{p-3} + \cdots$$

$$\therefore \quad \tfrac{1}{2}m^p(m+1)^p - \tfrac{1}{2}m^p(m-1)^p = {}^pC_1 m^{2p-1} + {}^pC_3 m^{2p-3} + \cdots$$

hence adding these for $m = 1, 2 \ldots n$

$$\tfrac{1}{2}n^p(n+1)^p = {}^pC_1 S_{2p-1} + {}^pC_3 S_{2p-3} + \cdots$$

i.e.

$$2^{p-1}S_1{}^p = {}^pC_1 S_{2p-1} + {}^pC_3 S_{2p-3} + \cdots$$

the last term being pS_{p+1} when p is even and S_p when p is odd.

With $p = 2$, 3, 4 this gives

$$S_1{}^2 = S_3, \quad 4S_1{}^3 = 3S_5 + S_3, \quad 2S_1{}^4 = S_7 + S_5.$$

By successive reduction any S_{2r-1} may be expressed as a sum of increasing powers of S_1 commencing with S_1^2, so that S_{2r-1}, as a polynomial in n, has the factor $n^2(n+1)^2$. Comparable, but more difficult, is the result that S_{2r} has the factor $n(n+1)(2n+1)$.

St. Albans School D. G. TAHTA

2754. A repeated integral

Let $K(a)$ be the mean distance between two independent points P, Q within the circle $|r| < a$.

$$K(a) = \frac{1}{(\pi a^2)^2}\iint_{\odot}\iint_{\odot} \sqrt{\{(x-\xi)^2+(y-\eta)^2\}}\,dx\,dy\,d\xi\,d\eta$$

$$= \frac{1}{(\pi a^2)^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{a}\int_0^{a} \sqrt{\{r^2-2rp\cos(\theta-\phi)+p^2\}}\,rp\,dr\,dp\,d\theta\,d\phi$$

$$= \frac{4}{\pi a^4}\int_0^{\pi}\int_0^{a}\int_0^{a} \sqrt{\{r^2-2rp\cos\psi+p^2\}}\,rp\,dr\,dp\,d\psi$$

$$= \frac{2}{15\pi a^4}\int_0^{\pi}\Big[\{(2-3\cos^2\psi)(r^4+p^4)-\cos\psi\,.\,rp(r^2+p^2)+4r^2p^2\}$$

$$\sqrt{\{r^2-2rp\cos\psi+p^2\}}+3\cos\psi\sin^2\psi\left\{p^5\sinh^{-1}\left(\frac{r-p\cos\psi}{p\sin\psi}\right)\right.$$

$$\left.+r^5\sinh^{-1}\left(\frac{p-r\cos\psi}{r\sin\psi}\right)\right\}\Big]_{r,p=0}^{r,p=a}d\psi$$

$$= \frac{4a}{15\pi}\int_0^{\pi}\Big[3\cos^2\psi-2+(3\cos\psi+4)(1-\cos\psi)\sqrt{(2-2\cos\psi)}$$

$$+3\cos\psi\sin^2\psi\left\{\sinh^{-1}\left(\frac{1-\cos\psi}{\sin\psi}\right)+\sinh^{-1}(\cot\psi)\right\}\Big]d\psi$$

$$= \frac{4a}{15\pi}\Big[\sin\psi\cos\psi+\sin\psi\left\{\sinh^{-1}\left(\frac{1-\cos\psi}{\sin\psi}\right)+\sinh^{-1}(\cot\psi)\right\}$$

$$-\tfrac{1}{3}(17+2\cos\psi-3\cos^2\psi)\sqrt{(2+2\cos\psi)}\Big]_0^{\pi}$$

$$= \frac{4a}{15\pi}\left(\frac{32}{3}\right)$$

$$= \frac{128a}{45\pi} \fallingdotseq 0{\cdot}9053\,a.$$

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2755. One-minute problem

Solve the equation

$$\frac{x}{x+1}-\frac{x(x-1)}{(x+1)(x+2)}+\frac{x(x-1)(x-2)}{(x+1)(x+2)(x+3)}-\cdots$$

$$+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{(x+1)(x+2)\ldots(x+n)}=\tfrac{1}{2}$$

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2756. The use of complex numbers in proving trigonometrical identities

Mr. J. C. Swaminarayan, in the Journal of the Indian Mathematical Society, June 1911, proposed the question:

Show that in any triangle ABC,

$$c^n = a^n \cos nB + n \cdot a^{n-1} \cdot b \cos \overline{(n-1)B - A} + \frac{n(n-1)}{2!} \cdot a^{n-2} \cdot b^2 \cdot \cos \overline{(n-2)B - 2A} + \ldots + b^n \cos nA.$$

n being a positive integer.

The solution of this is at once obvious if we observe that

$$c = ae^{iB} + be^{-iA} = ae^{-iB} + be^{iA} \qquad \text{(i)}$$

with similar expressions for b and a.

Formula (i) is merely the combined expression of the rule of sines and the rule of cosines.

From (i) $$c^n = \sum_{r=0}^{n} {}^nC_r \cdot a^{n-r} \cdot b^r \cdot e^{i[(n-r)B - rA]}$$

from which Mr. Swaminarayan's result follows. With the help of (i), we propose to solve several well known elementary examples.

(a) An immediate consequence of (i) is the law of cosines,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

for, $$c^2 = (ae^{iB} + be^{-iA})(ae^{-iB} + be^{iA})$$
$$= a^2 + b^2 + ab(e^{i(A+B)} + e^{-i(A+B)})$$

Since $$A + B + C = \pi, \quad e^{i\theta} = \cos\theta + i \sin\theta, \quad \text{and} \quad e^{i\pi} = -1,$$

therefore $$c^2 = a^2 + b^2 - 2ab \cos C.$$

(b) Now consider

$$(b + c)e^{iA/2} = ce^{-iA/2} + ae^{i(A+2C)/2} + ce^{iA/2}$$
$$= 2c \cos \frac{A}{2} + iae^{i(C-B)/2}$$

Equating real and imaginary parts we get

$$(b + c) \sin \frac{A}{2} = a \cos \left(\frac{C - B}{2}\right).$$

$$(c - b) \cos \frac{A}{2} = a \sin \left(\frac{C - B}{2}\right)$$

(c) Again consider

$$ae^{iA} + be^{iB} + ce^{iC}.$$

Using (i) for a, b, and c, we find

$$ae^{iA} + be^{iB} + ce^{iC} = \{be^{i(C+A)} + ce^{i(A+B)} + ae^{i(B+C)}\} + \{be^{i(C-A)} + ce^{i(A-B)} + ae^{i(B-C)}\}. \qquad \text{(ii)}$$

Using $A + B + C = \pi$, $e^{i\pi} = -1$, we get

$$2(a\cos A + b\cos B + c\cos C) = (be^{i(C-A)} + ce^{i(A-B)} + ae^{i(B-C)}),$$

and so
$$2\Sigma b\cos B = \Sigma b\cos(C-A)$$

and
$$0 = \Sigma b\sin(C-A).$$

But, if we group suitably the terms in R.H.S. of (ii) and use $2\cos\theta = e^{i\theta} + e^{-i\theta}$, we get

$$\Sigma ae^{iA} = \Sigma 2a\cos Ce^{iB}.$$

whence
$$\Sigma a(\cos A - 2\cos C\cos B) = 0$$
$$\Sigma a(\sin A - 2\cos C\sin B) = 0.$$

Lastly, consider

$$c^2(ae^{iA}) + a^2(be^{iB}) + b^2(ce^{iC})$$
$$= -bc^2e^{-iB} - ca^2e^{-iC} - ab^2e^{-iA}$$
$$+ c^3e^{i(A-B)} + b^3e^{i(C-A)} + a^3e^{i(B-C)}$$

or
$$\Sigma ca(ce^{iA} + ae^{-iC}) = \Sigma a^3e^{i(B-C)}.$$

Using (i) we get
$$3abc = \Sigma a^3e^{i(B-C)}.$$

and therefore
$$3abc = \Sigma a^3\cos(B-C),$$

and
$$0 = \Sigma a^3\sin(B-C).$$

A. K. RAJAGOPAL

2757. A triangle inequality

ABC is a plane triangle whose sides, opposite to A, B, C, have lengths a, b, c respectively. O is a point interior to the domain bounded by the triangle or on the triangle such that the distances of O from A, B, C are R_1, R_2, R_3 whilst the perpendicular distances of O from the sides BC, CA, AB are r_1, r_2, r_3 respectively. Erdős (1) suggested the following inequality

$$R_1 + R_2 + R_3 > 2(r_1 + r_2 + r_3).$$

This was proved by Mordell (2), (see (3)). The object of this note is to give a simple geometrical proof of the auxiliary inequality from which, following Mordell, Erdős's inequality can be established.

Let O' be the reflection of O in the internal bisector of angle BAC. Then O' is distant R_1 from A, r_3 from the side AC, and r_2 from side AB. The triangles ABO', ACO' do not meet except along their common side AO'. We have

$$\tfrac{1}{2}a \cdot R_1 > \text{Area of quadrilateral } ABO'C$$
$$= \text{Area } ABO' + \text{Area } ACO' = \tfrac{1}{2}r_2c + \tfrac{1}{2}r_3b$$

Hence
$$R_1 > r_2\left(\frac{c}{a}\right) + r_3\left(\frac{b}{a}\right),$$

and equality occurs if and only if AO' is perpendicular to BC, that is, if

$$\widehat{BAO} = \widehat{ABC} + \widehat{BAC} - \tfrac{1}{2}\pi.$$

By reflecting O in the internal bisectors of angles CBA and BCA we obtain

$$R_2 > r_1\left(\frac{c}{b}\right) + r_3\left(\frac{a}{b}\right),$$

$$R_3 > r_1\left(\frac{b}{c}\right) + r_2\left(\frac{a}{c}\right).$$

Equality occurs in the latter of these inequalities if and only if

$$\widehat{OBA} = \widehat{ABC} + \widehat{BAC} - \tfrac{1}{2}\pi,$$

Thus if equality occurs in the first and third inequality then $AO = BO$, and equality occurs in all three if and only if O is the circumcentre of triangle ABC. Adding these inequalities we obtain

$$R_1 + R_2 + R_3 > r_1\left(\frac{b}{c} + \frac{c}{b}\right) + r_2\left(\frac{a}{c} + \frac{c}{a}\right) + r_3\left(\frac{a}{b} + \frac{b}{a}\right).$$

Since $\frac{b}{c} + \frac{c}{b} > 2$ etc. we have Erdös's inequality

$$R_1 + R_2 + R_3 > 2(r_1 + r_2 + r_3)$$

and equality holds if and only if $a = b = c$ and O is the circumcentre of the triangle ABC.

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2758. A centre of gravity

The centre of gravity of a uniform sheet of metal in the form of a quadrilateral $ABCD$ can be found as follows:—

Let A_1A_2, B_1B_2, C_1C_2, D_1D_2 be the points of trisection of the sides AB, BC, CD, DA respectively, A_1 being nearer A than A_2 is and so on.

Complete the parallelogram $PQRS$ with sides along D_2A_1, A_2B_1, B_2C_1 and C_2D_1. Then G the point of intersection of the diagonals PR, QS of this parallelogram is the centre of gravity of the sheet $ABCD$.

Proof. If AC meets BD at O,

$$\triangle ABC : \triangle ADC = BO : OD,$$

and the uniform plate is equivalent to masses (proportional to) BO at A_1 and B_2 and masses (proportional to) OD at C_1 and D_2. Let the masses BO at A_1 and OD at D_2 be equivalent to x at P and y at S, and let SP meet AOC at T.

Since $PQ//AC//SR$ and $AD_2 = D_2D_1$ and $AA_1 = A_1A_2$, $SD_2 = D_2T$ and $TA_1 = A_1P$. Also since $SP//DB$, $D_2T : TA_1 = OD : BO$.

$\therefore$ moments about S give (proportionally)

$$2(BO + OD)\,x = OD \,.\, OD + (BO + 2\,OD)BO = (BO + OD)^2$$

$$\therefore \qquad x = \tfrac{1}{2}(BO + OD) = y.$$

and the plate is equivalent to equal masses at P, Q, R and S, so the centre of gravity of the plate is at G.

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2759. Tuckey's proof of Feuerbach's theorem, and yet another

Feeling Mr. Tuckey's charming article for work-shy geometers (*Gazette* XXXIX, 31) to be somewhat disfigured by an ugly proof that $s^2\Delta^2 . IN^2$ is a polynomial in a, b, c, I have been using mean centres to deal with this step, and the experience has been instructive.

Since I is the mean centre of loads a, b, c at A, B, C

$$\Sigma(a . NA^2) = 2s . NI^2 + \Sigma(a . IA^2).$$

and since $s . IA^2 = bc(s - a)$, it is sufficient for Mr. Tuckey's argument if we prove that $s\Delta^2 . NA^2$ is a polynomial. For this purpose it is natural to look for a triangle whose vertices are at known distances both from N and from A. Such a triangle is the mid-point triangle $A'B'C'$, which has N for its circumcentre. Since the sum of the loads $a \cos A$, $b \cos B$, $c \cos C$ required at A', B', C' is $2\Delta/R$,

$$R\{AA'^2a \cos A + AB'^2b \cos B + AC'^2c \cos C\} = 2\Delta(NA^2 + \tfrac{1}{4}R^2), \quad (1)$$

and we have Δ^2NA^2 expressed in terms of $R^2\Delta^2$, which is a multiple of $a^2b^2c^2$, of $A'A^2$, $B'A^2$, $C'A^2$, which are multiples of $2b^2 + 2c^2 - a^2$, b^2, c^2, and of the three products such as $R\Delta \, a \cos A$, which is a multiple of $a^2(b^2 + c^2 - a^2)$.

Having reached this point I perversely tackled the unnecessary algebra, to find that to carry it through in detail was a serious matter; I could make an efficient job of it, but not an elegant or a quick one. Then I wondered what difference it would make if I used instead of $A'B'C'$ the triangle $DB'C'$, where D is the projection of A on BC. The change seems almost trivial, for $DB'C'$ is only a reflection of $A'C'B'$, and the same set of loads is wanted in the location of N. Nevertheless, the effect is sensational. Instead of (1) we have

$$2\Delta(NA^2 + \tfrac{1}{4}R^2) = R\{DA^2a \cos A + C'A^2b \cos B + B'A^2c \cos C\}, \quad (2)$$

implying

$$2\Delta a(4NA^2 + R^2) = R(16\Delta^2 \cos A + abc(c \cos B + b \cos C))$$

that is

$$a(4NA^2 + R^2) = 8R\Delta \cos A + 2R^2a,$$

and since $\Sigma \cos A - 1 = r/R$, we have now

$$2\Sigma(a . NA^2) = s(4r^2 + 4rR + R^2),$$

and therefore, since $\Sigma(a . IA^2) = abc = 4srR$, finally

$$4NI^2 = 4r^2 + 4rR + R^2 - 8rR = (R - 2r)^2.$$

It should be added that once we associate N with the triangle $DB'C'$ we can find NA^2 without treating N as a mean centre, for the reflection N_1 of N in $B'C'$ is the circumcentre of $AB'C'$. Since

$$AN^2 - AN_1^2 = AD . N_1N,$$

we have immediately

$$NA^2 = AD . OA' + \tfrac{1}{4}R^2,$$

a simplified version of (2).

The Copse, Sonning-on-Thames. E. H. NEVILLE

1908. The same small δ that represents mathematically the consideration in which an Irishman holds the British Government.—K. Scoresby Routledge, *The Mystery of Easter Island* (1919). [Per Mr. J. C. W. De la Bere.]

CLASS ROOM NOTES

1. The quadratic equation $x^2 - px + q = 0$

To solve this equation graphically, plot the points (0, 1) and (p, q) on squared paper. Draw the circle with these two points as diameter. Then this circle meets the x-axis in the points whose x-coordinates are the roots of the equation. The proof may be by circle geometry, or the method may be used to introduce or exemplify two ways of finding the equation of a circle on a given diameter.

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2. The volume of a sphere

Consider a circular cylinder whose height is equal to the diameter. In it place a double cone, with bases on the ends of the cylinder and with vertex at the centre of the cylinder. Also imagine a sphere fitting inside the cylinder. It is easy to establish that for any section of the three solids, taken perpendicular to the axis of the cylinder, the area of the section of the sphere equals the area of the section of the cylinder less the area of the section of the cone. Assuming the formulae for the cylinder and the cone, this allows the formula for the volume of the sphere to be deduced.

(Archimedes held this and his theorem on the surface area of the sphere to be his greatest discovery.)

J. W. HESSELGREAVES

3. Three Geometry Notes

1. *Another Angle Notation.*

In 2 recent notes, (2462 and 2621), writers have suggested new variants of the single-letter angle notation. The method recommended below combines many of the advantages of both these, as well as incorporating useful extensions. Personally, I have no objection whatever to using Greek letters for angles, yet I realize that some teachers will consider the need to use this extra symbolism detrimental to Mr. Cossins' excellent suggestions.

Why not, then, use numbers as suffixes? Thus, in Fig. 1,

$\widehat{DAC} = A_1$, $\widehat{CAB} = A_2$, and $\widehat{DAB} = A_{12}$ (read as: 'A-one-two').

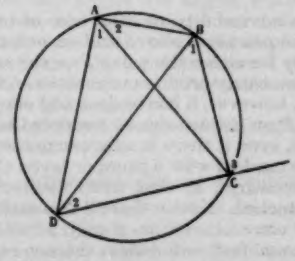

FIG. 1

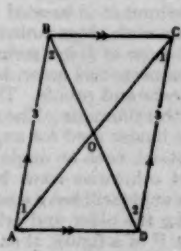

FIG. 2

This still enables one to mark equal angles with the same symbol (now a number), for this, of course, is the great merit of this type of notation. As an

example, the cyclic quadrilateral theorem exhibited in Fig. 1 is $C_3 = A_{12}$, and the proof, sketched briefly, becomes:

$$A_1 = B_1;\ A_2 = D_2;\ C_3 = B_1 + D_2 = A_1 + A_2 = A_{12}.$$

Since there are few normal geometry proofs in which one needs to mark even as many as 6 pairs of angles equal, one is never likely to have an angle with a suffix *twelve* which could be confused with the A_{12} above. As with Mr. Cossins' method, if 2 equal angles have a common vertex P, we mark them, for example, 4 and 4′, and write $P_4 = P_{4'}$. To forestall the expected criticism, it is agreed that there *is* some slight danger of confusion between $P_{4'}$ and P_{41}, though it can be claimed that occasions when both symbols occur in the same question will be rare, and that the students who confuse them when they do will probably be fewer than the ones who already confuse, say, angles ZXY and ZYX!

An extra advantage claimed for this method is that in addition to marking equal *angles* on the diagram with the same number, equal *lengths* can also be so marked, the number, in this case, being written *across* the line segment in question, and not written alongside as it would be (together, one hopes, with some units) if it were a measurement of length. This at least produces an orderly scheme, for, having eliminated the crosses, the double rings, and the higher "rainbows" with which people decorate their equal angles, one may as well systematize the whole operation by eliminating also the miscellaneous grids and twiddles which they use to mark equal lengths.

There is one further point which teachers should note. Suppose one is tackling a congruent triangle proof of the simplest kind, involving just one pair of triangles, e.g. proving that the diagonals of a parallelogram bisect each other (Fig. 2).

In the triangles AOB, COD,

$$(1)\ A_1 = C_1 \qquad (2)\ B_2 = D_2 \qquad (3)\ AB = CD, \quad \text{etc.}$$

If the diagram and proof are arranged in this way, one can not only secure the essential agreement between the numbers marked on the angles and the associated suffixes, but also an optional correspondence between the numbers marked on the 3 relevant pairs of parts and the enumeration (1), (2), (3), of the corresponding statements in the proof. It is not necessarily recommended that one should require one's students to regiment their own work in this way, particularly since it will not be possible to follow this plan consistently once the simplest examples are past. Teachers, however, may find it a helpful eye-catcher to reinforce their blackboard work in the early stages of formal geometry.

Some comment is needed on the advisability or otherwise of omitting all angle signs with these notations (as practised above and in notes 2619 and 2621). The use of P indiscriminately for either the point P or the angle P can scarcely cause actual error, but it can easily produce vagueness and obscurity in statements and proofs. There is, however, a convenient and simple way of avoiding this dilemma. The use of P on its own can be restricted to the point P, and no longer used for *any* angle, even if there is only one non-reflex angle at P. Instead, such an angle can be marked with a number (even though this would not otherwise have been necessary), so that when mentioned in the proof, the angle will have a suffix attached. Notice that all this really involves is replacing the older and arbitrary convention that, if only 2 half-lines meet at a point P of a figure, then $\hat{P}$ (unqualified) will denote the non-reflex angle, the other being specially described,* with the more methodical convention

* This was a convention, incidentally, which always caused trouble when the angle concerned could be either greater or less than a straight angle according to how the diagram was drawn (as with angles subtended by general arcs at the centre of a circle).

that, *whichever* of the 2 angles is intended, some suffix or other will be added to distinguish it, and the corresponding reference added to the diagram.

There is one type of proof in which single letters are often used which requires special mention. As an illustration, the reader is referred to the proof of the cyclic quadrilateral theorem given as example 3 (appendix 2) on p. 140 of the Second Geometry Report. Here, p, q, r, s denote certain pairs of equal angles (though the present writer shares many people's dislike of this use of small roman letters for angles). None of the new suffix notations is noticeably better adapted to this kind of proof than the old, but once it is realized that the solution given is basically algebraic, this type can always be dealt with by writing something like "Let $A_1 = D_1 = p°$", and treating p as a *number*, rather than an angle. (Incidentally, the proof given in this example can be criticized as being one of the least elegant of all the possible proofs of this theorem. For, unlike most others, it has to be modified when O lies outside the quadrilateral.)

The habit of numbering the equal parts of a geometry figure 1, 2, 3, ... in an orderly fashion, is one which I was taught as a boy at Bradford G.S. by Mr. A. C. Baggley; the P_{mn} idea, which finally eliminates the need for any 3-letter angle notation, I owe to the stimulus of reading Mr. Cossins' note.

2. *A Plea for the Straight Angle.*

May I, diffidently, urge that the neglected straight angle be admitted freely into respectable and regular usage, and that it even be considered the fundamental unit of angle, in preference to the right angle? The reasons for this novel break with tradition will, it is hoped, become clear as this article progresses. The development below is of the quasi-rigorous type suitable for the presentation of fundamental notions at the elementary level; it parallels the usual treatment and ignores the questions involved in the axioms of continuity, etc. It starts by supposing that the concepts of angle, arm and vertex are already understood.

Formal Definition. A straight angle is an angle whose arms are parts of the same line and lie on opposite sides of the vertex.

Postulate. All straight angles are equal.

Other Definitions.

1 revolution = 2 straight angles.

1 right angle = $\frac{1}{2}$ straight angle.

Supplementary angles are angles whose sum is equal to 1 straight angle.

Complementary angles are angles whose sum is equal to 1 right angle.

Acute, obtuse and reflex angles are defined (as they should be) from their inequality relations with the right angle, straight angle and revolution, without mentioning the less fundamental notion of the degree at all.

$AO \perp POQ$ is defined as an alternative way of writing the statement "AO bisects the straight angle POQ."

Adjacent angles† are 2 angles with a common vertex and arm which together make up 1 straight angle.

Theorems. Adjacent angles are supplementary.

If 2 supplementary angles have a common vertex and arm, their remaining arms are parts of the same line.

Vertically opposite angles are equal.

If $AB \perp CD$, then $CD \perp AB$.

† The writer does not subscribe to the school of thought which defines *any* 2 angles with a common vertex and arm as adjacent, and then has to write "adjacent angles on the same line" for the only case which really requires a reference.

Notes on the advantages of this treatment.

1. It is easier to put into words a definition of the straight angle than of either the revolution or the right angle.

2. The postulate that "all straight angles are equal" seems conceptually more fundamental than Euclid's postulate that "all right angles are equal" though this subjective feeling may not be shared by others. Logically, of course, either approach is equally satisfactory, provided the appropriate sequence is maintained.

3. The definitions chosen make the fact that adjacent angles are supplementary appear as a simple corollary, rather than an additional postulate.

4. Instead of many elementary proofs that $AO \perp POQ$ finishing like this:

$$O_1 = O_2$$

But O_1 and O_2 are supplementary. (Adjacent angles).

$\therefore O_1$ and O_2 are each right angles

$\therefore AO \perp POQ$

all that we now have is:

$$O_1 = O_2$$

$$\therefore AO \perp POQ.$$

5. The construction for drawing a perpendicular to a given line at a given point on that line (the method usually taught first) no longer needs the separate mention it still gets in many textbooks. It is exactly the same construction as that used for bisecting an angle, and since we are now unashamedly *trying* to bisect a given straight angle, rather than to find a line which makes a pair of adjacent angles equal—a subtle difference—the proof, also, is now identical with that for the general case.

6. "The sum of the angles of a triangle is equal to 1 straight angle." This should not be *enunciated* as "... 2 right angles" any more than it should be as "... 180°," for right angles do not occur naturally in any of the proofs of this theorem. What is shown is that, as a consequence of the Euclidean postulates, the 3 angles of the triangle are equal to certain others which are so situated that together they make up a straight angle. (The concept of straight angle is equally relevant in the non-Euclidean geometries, and what is proved there is that, in the elliptic case, the sum of the angles of a triangle is greater than 1 straight angle, while in a hyperbolic geometry, the sum is less than 1 straight angle; again, right angles don't enter into the picture in the first instance.)

7. "The sum of the interior angles of an n-sided polygon is equal to $(n - 2)$ straight angles."

The formula now corresponds more nearly to the method of proof. Assume that the polygon is convex or "not-too-concave" (the reader knows what is meant!), so that each vertex can be joined to a point P by line segments interior to the polygon. If P lies (1) at a vertex, there are $(n - 2)$ triangles, (2) on an edge, but not at a vertex, there are $(n - 1)$ triangles, (3) at an interior point, there are n triangles. In the various cases, the sums of the angles of the triangles are equal to $(n - 2)$, $(n - 1)$, and n straight angles. On subtracting 0, 1, or 2 straight angles respectively for the unwanted angles at P, the formula is obtained naturally, without "right angles" or "$2n$" appearing. (It is fully realized that proof (2) is less useful than (1) or (3); it was added merely for completeness.)

If, on the other hand, the polygon is "violently" concave, it can still be divided by diagonals (not now concurrent) into $(n - 2)$ triangles, and this, of

course, gives the best general proof, though not necessarily the one which should be taught at a first reading.

8. The fact that the fundamental conversion formula is π radians = 1 straight angle is another, though absurdly trivial, reason for suggesting that the straight angle be used more.

9. A final way of looking at the matter, perhaps, is to suggest that the extent to which the straight angle is more fundamental than the right angle has a close analogy in the much greater frequency with which we need to refer to supplementary angles compared with complementary angles.

The decision which to make the more fundamental is a delicate and possibly unimportant one, but in any case the neglect of the straight angle seems unwarranted, and it certainly should take its place alongside the right angle, without always being replaced by "2 right angles."

3. *The Definition of Locus.*

"A locus is the path of a point which moves according to certain given conditions." So say almost all the textbooks; but is this a good definition? Is it even an accurate description of what a locus is?

Many students find the general idea of locus difficult, even if they can do the ones about the locus of the tip of the man's nose as the lift goes up and down. In the writer's opinion, the above definition does not make it any easier for our pupils. The heretical view is here proposed that the importance of the moving point idea has been seriously exaggerated.

Now it is not denied, of course, that in the early informal stages (after we have all finished digging for the buried treasure) the idea is immensely valuable, nor is it denied that many loci are usefully thought of in this way; consider merely the cycloid and other roulettes, and the conics defined by the focus-focus or even focus-directrix properties.

The criticism made is that this naive approach is persisted in too long at the systematizing stage. For a locus may be either a single surface, a finite or infinite number of surfaces, a single curve, a finite or infinite number of curves, a single point, a finite or infinite number of points, any combination of these, or the locus may not be realizable at all. It is only in the third and occasionally the fourth of these cases that the dynamic idea is really helpful. If a point is 2 in. away from one line and 3 in. away from an intersecting line, its locus in the plane is 4 points, and a pupil who has been too thoroughly indoctrinated with the path idea is understandably confused. Even surfaces give trouble to a student who, very naturally, thinks of a path as a continuous *curve* (i.e. a *one*-parameter family of points), and when he has to find the locus of a point 2 in. away from one line and 3 in. away from a parallel line, which does not in general contain *any* points, it is not surprising if he gives the whole topic up.

Our definition should have been sufficiently general to cover all cases, so consider the following alternative:

"A locus is the set of all points which satisfy certain given conditions."

At one stroke, all these particular difficulties disappear, for although the concept of "set" may be a nightmare to a mathematical philosopher, it is doubtful if a fourth-former has to think twice before he understands its meaning here, and it certainly should give less trouble than the path approach. The set of points may lie on surfaces or curves, it may contain isolated points or it may be what we should call the empty set.

The double proof required to justify a locus statement now involves proving that every point which satisfies the conditions belongs to the set and that every point which belongs to the set satisfies the conditions. This, after all, is more closely akin to what is actually done in a locus proof. The points dealt with there are thought of as typical members selected from the set of all the

possible points, not as point planets pursuing certain inexorable orbits.

If the dynamic concept is to be soft-pedalled, it will mean that in the later stages many questions now phrased as "What is the locus of a point which moves so that it is ...?" will be better approached as "What is the locus of a point which is ...?"

It might be argued that the path notion is fruitful in analytic geometry. But no good teacher will allow his students to start by writing "Let (x, y) be a moving point on the locus." He will insist instead on "Let (α, β) be a *fixed* point belonging to the locus," followed by obtaining the *condition which* α *and* β *must satisfy*, and only changing to coordinates (x, y) at the very end by writing "This condition expresses the fact that (α, β) lies on the curve ..., which is therefore the required locus."

One final point. The new definition is more in keeping even with the etymology of the word locus as meaning "place" or "*lo*cation"; the locus can be thought of as the set of all the places where the point can possibly be found when it satisfies the conditions.

ROGER F. WHEELER

4. Radius of curvature in Polars

The following may appeal to those who find the usual textbook treatment somewhat shapeless. With the usual notation, $x + iy = re^{i\theta}$
Differentiate twice,

$$\frac{dx}{d\theta} + i\frac{dy}{d\theta} = \left(\frac{dr}{d\theta} + ir\right)e^{i\theta} \qquad \ldots(1)$$

$$\frac{d^2x}{d\theta^2} + i\frac{d^2y}{d\theta^2} = \left[\left(\frac{d^2r}{d\theta^2} - r\right) + 2i\frac{dr}{d\theta}\right]e^{i\theta} \qquad \ldots(2)$$

and putting $-i$ for i in (1)

$$\frac{dx}{d\theta} - i\frac{dy}{d\theta} = \left(\frac{dr}{d\theta} - ir\right)e^{-i\theta} \qquad \ldots(3)$$

Multiplying corresponding sides of (2) and (3) and (1) and (3) in turn, we get

$$\frac{d^2y}{d\theta^2}\cdot\frac{dx}{d\theta} - \frac{d^2x}{d\theta^2}\cdot\frac{dy}{d\theta} = 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} + r^2$$

and

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$$

whence the polar form follows at once from the parametric form.

Haberdas'ers' Aske's Hampstead School V. I. TODHUNTER

5. A Paradox: or what has Achilles taught us

The scholarship question, which read: "A spherical ball is released from rest at a height h above the horizontal ground. The coefficient of restitution is e (< 1). Find the time that elapses before the ball remains at rest on the ground" prompted these musings. Since there is no last bounce and after every bounce there is a residual velocity, be it never so small, how does the ball come to rest?

V. I. T.

6. Calculation

Involved and tedious arithmetical calculations have largely disappeared, but a modern boy's scanty acquaintance with pure numbers may lead to his unawareness of their content. Tables may be learnt as a drill, but the weakness appears when finding, for example, an HCF or LCM. A remedy might lie in improving his basic equipment by, say, learning and using the squares of the first twenty integers, the rules for divisibility by 3 or 11, the factors of familiar numbers such as 1760, 2240, 4840. Imagine trying to solve a geometry problem based on the circle without a good knowledge of and practice in applying the properties connected with that figure.

Ratcliffe College, Leicester L. G. HURDIDGE

7. Chord of a conic with a given mid-point

The chord P_1P_2 of the conic $S = 0$ is $S_1 + S_2 - S_{12} = 0$, since it is linear and P_1, P_2 lie on it. Let P_3 be the mid point, then $S_3 = \frac{1}{2}(S_1 + S_2)$, and the equation of the chord becomes

$$S_3 - \tfrac{1}{2}S_{12} = 0$$

i.e. $S_3 - \text{constant} = 0$

giving $S_3 - S_{33} = 0$ as P_3 lies on it.

Alternatively $S_{33} = \frac{1}{2}S_{12}$ may be proved algebraically.

$$S_{33} = a\left(\frac{x_1 + x_2}{2}\right)^2 + \dots$$

$$= \tfrac{1}{4}(S_{11} + 2S_{12} + S_{22})$$

$$= \tfrac{1}{2}S_{12} \text{ as } S_{11} = S_{22} = 0$$

For a central conic, centre P_0, it is interesting to proceed as follows. The diameter P_0P_3 meets the conic in $\lambda P_0 + \mu P_3$, where $\lambda + \mu = 1$. The tangent at this point is $\lambda S_0 + \mu S_3 = 0$
i.e. $S_3 + \text{constant} = 0$ as $S_0 = \text{constant}$.
This is parallel to the chord, so the equation of the chord is

$$S_3 + \text{constant} = 0$$

i.e. $S_3 - S_{33} = 0$ as P_3 lies on it.

Ratcliffe College, Leicester A. L. DAVIES

8. The shortest distance between two skew lines

On a piece of squared paper place the origin O centrally and draw axes to the four points of the compass. Mark the North point Z, the West point X, the East and South points both Y. Cut the paper along the East axis and fold up to make mutually perpendicular planes. Draw the lines $y = 0$, $x + z = 4$ and $z = 0$, $2y = x + 6$. Mark the point $(p, 0, 4 - p)$ as P. Mark the point $(q, \frac{1}{2}(q + 6), 0)$ as Q and write down the value of PQ^2. Using partial differentiation find the minimum value of PQ^2. Check the perpendicularity of PQ with the skew lines by using direction cosines.

J. W. HESSELGREAVES

9. The cosine rule

(i) In Note 2556 (Sept. 1955), P. Gant gave a proof based on the formula $a = b\cos C + c\cos B$. The following variant has the advantage of starting on the a^2 side.

$$\begin{aligned} a^2 &= a(b\cos C + c\cos B) \\ &= ba\cos C + ca\cos B \\ &= b(b - c\cos A) + c(c - b\cos A) \text{ etc.} \end{aligned}$$

(ii) But at this stage in the course the pupils still really need a figure including an altitude from B or C. In *any* figure containing the altitude from B, Pythagoras' Theorem gives

$$a^2 = (b - c\cos A)^2 + (c\sin A)^2 \text{ etc.}$$

This seems preferable to the juggling with p (altitude) and x (projection) that often goes on.

(iii) It also suggests an amusing refinement, in which we dispense with the figure.

$$\begin{aligned} a^2 &= a^2(\cos^2 C + \sin^2 C) \\ &= (a\cos C)^2 + (a\sin C)^2 \\ &= (b - c\cos A)^2 + (c\sin A)^2, \text{ etc.} \end{aligned}$$

Jordan Hill Training College, Glasgow A. G. SILLITTO

10. The length of a perpendicular

J. B. M.'s method (Note 2575, December 1955) is well enough, but I think we should take advantage of the fact that this is one of the occasions when we can show the power of a simple transformation—in this case, the shift of origin.

The length of the perpendicular from the origin to the line $ax + by + c = 0$ can be found in various ways. For example:—

(i) Using J. B. M.'s method, with (x_2, y_2) the foot of the perpendicular, we get

$$bx_2 - ay_2 = 0$$

and

$$ax_2 + by_2 = -c,$$

from which $x_2^2 + y_2^2$ is found by solving, or, more simply, by squaring and adding.

(ii) Finding the intercepts on the axes, we can obtain the altitude of the triangle formed by the axes and the given line, from its area; or use proportionalities from the similar triangles in this figure.

(iii) By a variety of methods, we can find r if the circle $x^2 + y^2 = r^2$ touches the given line.

Thereafter, a simple shift of origin to (x_1, y_1) replaces the c in the equation of the line, and hence in the numerator of the length formula, by $ax_1 + by_1 + c$. In general, while pupils are learning their way about in coordinate geometry, it behoves us to keep geometrical ideas in the forefront, and make use of the light they shed on the algebraic manipulation.

In fact it is desirable to return to the matter later on, and discuss it by way of a rotation of the axes which brings OX into coincidence with the perpendicular. The equations quoted in (i) then acquire a deeper significance, when we write down the expressions for X_2, Y_2; while if we express x_2 and y_2 explicitly in terms of X_2, Y_2, the "squaring and adding" device is seen to be related to the invariance, under the transformation, of the form $x^2 + y^2$.

A. G. SILLITTO

11. The quadratic equation: $ax^2 + bx + c = 0$

Obviously, if the trinomial factorises at sight, there is no better method. If not, one "completes the square" or uses the formula (obtained by completing the square). The process of completing the square admits of step-by-step geometrical illustration, at least in the case of c/a negative.

I suggest that there are other methods, of greater outlook value for the embryo mathematician. Each is discussed here in general terms; prior practice of the method with numerical coefficients will be necessary in class.

(1). This involves simply a change of language. Instead of proposing to complete a square, with its probably irrelevant geometrical reference, one seeks a linear substitution which will transform the equation into a simple form—with zero coefficient of the first degree term. In general, set $x = y + k$ and obtain $ay^2 + (2ak + b)y + ak^2 + bk + c = 0$. Choosing $k = -b/2a$, the result follows.

This method looks forward to the solution of the cubic equation; and, less remotely, to the powerful device, often unduly neglected or postponed in schools, of change of origin in coordinate geometry.

(2). Write the equation in the form $ax(ax + b) = -ac$. In fact, any quadratic states the *product* of two numbers whose *difference* is known. We aim to find first the number mid-way between these two—viz., $y = ax + b/2$. This yields at once $y^2 - b^2/4 = -ac$; etc.

This method, like (1), can be related to a graphical treatment, and draws attention to the symmetry about a certain vertical axis of the graph of the function $ax^2 + bx + c$.

(3). This method makes use from the outset of the sum and product of the roots of the quadratic equation, and assumes familiarity with a certain system of simultaneous equations. (These are not insuperable obstacles.) It is perhaps desirable to proceed less formally than in the chapter on "Theory of Quadratics", so, referring back to the solution by factors, we find that the problem consists of finding two numbers, p and q, whose sum and product are $-b/a$ and c/a.

i.e.
$$p + q = -b/a$$
$$pq = c/a.$$

We know that we can find the difference $p - q$ ($p > q$ supposed) from $(p + q)^2 - 4pq$, and obtain

$$p - q = \sqrt{b^2 - 4ac}/a; \text{ and hence } p \text{ and } q.$$

These methods may be regarded as alternatives to completing the square, for numerical equations; *or* as alternative ways of reaching the general formula.

A. G. Sillitto

12. Pythagoras' theorem and its converse

In the second report on the Teaching of Geometry the following proof of the theorem of Pythagoras is given.

Given $\angle BAC = 1$ right-angle, to prove $BC^2 = AB^2 + AC^2$

Construction. With centre C and radius CA describe semi-circle DAE.

Proof. Let the angles which BD, DC, CE subtend at A be called A_1, A_2, A_3 respectively. Since BA is perp. to CA a radius.

$\therefore$ BA is a tangent and $\angle A_1 = \angle E$

Also $\angle B$ is common in Δs BAD, BEA,

$\therefore$ Δs are equiangular and similar,

$\therefore$ $\dfrac{BA}{BD} = \dfrac{BE}{BA}$,

$\therefore$ $BA^2 = BD \,.\, BE = (BC - CD)(BC + CE)$

$= BC^2 - CA^2$

$\therefore$ $BC^2 = AB^2 + AC^2$.

A proof of the converse theorem independent of the theorem of Pythagoras, using the same figure and construction.
Given $BC^2 = AB^2 + AC^2$, to prove $\angle BAC = 1$ right-angle.

Proof. $AB^2 = BC^2 - AC^2$

$= (BC - AC)(BC + AC) = (BC - CD)(BC + CE)$

$= BD \,.\, BE$

$\therefore$ $\dfrac{AB}{BD} = \dfrac{BE}{AB}$ and $\angle B$ is common in triangles BAD, BEA.

$\therefore$ the sides about the common angle are proportional

$\therefore$ Δs BAD, BEA are similar

$\therefore$ $\angle A_1 = \angle E = \angle A_2$ ($CA = CE$ radii).

Add $\angle A_2$ to each.

Then $\angle BAC = \angle DAE = 1$ right-angle ($\angle$ in semi-circle).

Hawarden Grammar School JAMES BELL

13. The derivative of a quotient

In most elementary calculus books the formula for the derivative of a quotient is given as:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

This works well in differentiating functions like $\dfrac{x}{1+2x}$ but involves unnecessary labour when differentiating functions like $\dfrac{3x-2}{(5x+1)^3}$ or $\dfrac{2x-3}{\sqrt{x^2+1}}$.

I have found that boys obtain results more quickly and more accurately using the formula

$$\frac{d}{dx}\left(\frac{u}{v^n}\right) = \frac{v\dfrac{du}{dx} - nu\dfrac{dv}{dx}}{v^{n+1}}$$

For example

$$\frac{d}{dx}\left(\frac{5x-3}{\sqrt{x^2+1}}\right) = \frac{5(x^2+1) - \frac{1}{2}\,.\,2x(5x-3)}{(x^2+1)^{3/2}} = \frac{3x+5}{(x^2+1)^{3/2}}$$

King Edward's School, Birmingham. G. COOPER

REVIEWS

"O" Level Tests in Arithmetic. By C. C. T. Baker: Pp. 144; 7s. 6d. 1956. (Methuen)

"O" Level Tests in Algebra. By J. Hislop. Pp. 127; 7s. 6d. 1955. (Methuen)

Graded Examples in Elementary Geometry. Pp. 117; 1956. (Kenion Press)

"O" Level Tests in Arithmetic contains considerably more than its title indicates; the book is a concise text-book, each topic being discussed briefly but adequately in language intelligible to the average pupil of a Secondary School. Explanations are followed by a set of worked examples of graded difficulty and a selection of exercises to be attempted. Thus the chapter on "Rates and Taxes" has nine worked examples following a page of explanation, and finishes with 31 exercises. The worked examples range over Local Government Finance, the cost of running a house, and Income Tax, the examples on the last being commendably clear. This lay-out is very suitable for the student working by himself; he will find all he needs in Arithmetic at this level.

It is important that the worked examples are accurately set out, particularly for the above-mentioned student; a few misprints have been found in these (e.g. p. 5, Ex. 3, line 6; p. 8, Ex. 15, line 14; p. 65, Ex. 4, line 7; p. 77, Ex. 10, where $3\frac{1}{4}\%$ should be $2\frac{3}{4}\%$). In the worked example on page 19 (Ex. 17) the question mentions 1 mile as being equivalent to 1·609 km, whereas in the working 1 inch is used as 2·54 cm. In the graph on page 105, the points are plotted without the small circles on which the author insists in the construction on the previous page.

The exercises are actual G.C.E. questions or are modelled on such lines; answers are provided. The book can be recommended to any teacher searching for such exercises, of which there never seem to be sufficient.

In **"O" Level Tests in Algebra,** the first section is devoted to exercises on the main topics of the "O" level syllabus. The rest of the book contains 30 test papers, each divided into sections A and B. There is no explanatory matter or worked examples, but this does not necessarily detract from its value, as the main need of many teachers is for exercises only.

All the important topics are covered, the different requirements of the G.C.E. bodies and that of the Scottish Leaving Certificate being considered. The exercises on graphs are rather pedestrian and could have included cubics or the graph of $\log x$, which are included in some syllabuses. The introductory exercises might with advantage have been more extensive with extra simple routine exercises for the weaker pupils, who would find the section on "Changing the Subject" rather fast-going. For the average pupil however this is a very sound and interesting collection, attractively set out and printed, and well worthy of inspection by the Secondary School teacher.

Graded Examples in Elementary Geometry was originally prepared for use at Eton College to provide a course leading to G.C.E. "O" level. It consists of exercises grouped in 29 topics ranging from scale drawing to the properties of similiar triangles, taking in all the usual G.C.E. facts en route. The exercises are followed by three appendices; appendix I gives a list of theorems used; those whose proofs are commonly asked in examinations are written out in appendix II; in appendix III diagrams of the simpler constructions are written out, a useful feature of the book.

The exercises have been carefully chosen and are well graded. There are a large number of numerical examples such as occur in Section A, some requiring

simple trigonometry and arithmetic (e.g the total length of the band passing round two pulleys). There is an unusually interesting section on the sphere and cone; while these exercises reduce to practice on the circle, they form a refreshing change. The proof of Pythagoras is the usual one by areas and congruent triangles and could conveniently have been superseded by that using similiar triangles; there is no proof of the "Extension of Pythagoras", here called "the cosine formula", but the rectangle properties of the circle (proved by similiar triangles) and the angle bisector theorems are given.

As a supplement to a full text book this is an excellent set of exercises.

E. H. COPSEY.

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The examples are arranged in such a way that the book can serve as a valuable introduction to Geometry, with the teacher developing his theory of the subject as he progresses.

Every effort has been made to apply theory to practical problems, and the diagrams are pleasingly realistic. Pupils can see the ladder leaning against the windowsill, the football net, the ship navigating a passage. This is particularly the case in the early stages, when a learner often experiences difficulty in transfer from reality to a geometrical diagram.

It is a good book, well thought out, well presented, and it will be found useful.

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The book is based on sound principles and a measure of integration is achieved between the various branches of the subject. No work on the circle apart from its mensuration is attempted and it remains to be seen what standards are to be set in formal geometry. These are not easily maintained especially, perhaps, in a textbook of general mathematics, where the style and flavour of the subject can be missed in the "mixture," and a somewhat better layout is sometimes desirable in the present volume.

W. FLEMMING.

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The order is unusual and pleasing, for it leads on from the ideas which the pupils have gained in the ordinary level Syllabus B mathematics course. There are many varied, well-graded exercises for the pupil, who receives much help from the clear explanations of the principles involved and the worked examples. The diagrams throughout are good and the general presentation attractive.

N. BARTON.

The Theory of Suspension Bridges. By Sir ALFRED PUGSLEY. Pp. 136. 42s. 1957. (Edward Arnold)

A schoolboy, on holiday in Wales, stands rapt in wonder, absorbed in Telford's Menai suspension bridge. Some years later, after being engaged himself on research and advisory work on suspension bridges, he produces this most readable book. In it the diverse contributions of Bernouilli, Brunel, Rankine, Navier, Melan, Routh, Steinmann, Southwell, etc., are critically reviewed and connected together. The result is this small volume which brings out by logical argument the main characteristics of suspension bridges in a way that the schoolboy of today who has done sixth form mathematics or mechanism should to a large extent find intelligible, while those who have themselves worked on suspension bridge theory will be charmed with the neat presentation of what could easily be tedious analysis.

The book opens with a historical review of suspension bridges, of their development and of the failures which have occurred, including the 1940 disaster to the Tacoma Narrows suspension bridge. It closes by explaining how wind-excited oscillations arise and what can be done to reduce the possibility of their occurring. In between the story of the growth of knowledge of suspension bridge theory is unfolded.

Sir Alfred starts his analysis by considering the geometrical form of a cable. Taking 1,000 feet as the maximum practicable height of towers, he shows that the greatest possible distance which can be spanned is about 10 miles, some six or seven times that proposed for the new Severn Bridge. He then shows that the dead weight of the cable provides the "stiffness" which restricts the deflexions which would occur when external loads are applied to a simple cable. Further stiffness is provided in practice by longitudinal girders which carry the roadway. Sir Alfred introduces first Rankine's analysis (1858) of this problem. This assumes that the girder distributes concentrated live loads uniformly over the whole span. He then shows its connection with Navier's elastic theory (1826) in which the intensity of the uniformly distributed load is chosen so that the strain energy stored in the whole structure is a minimum. Melan's deflexion theory (1888), in which no such assumption is made, is next developed and gives a fourth-order differential equation. This becomes sensibly linear when external live loads are small compared with the bridge weight. The equation is then analogous to the equation of a tie subjected to lateral load. By using "influence coefficient" ideas, which he himself developed, Sir Alfred replaces this equation with a set of linear simultaneous algebraic equations. Timoshenko's use of Fourier series in solving the differential equation and Southwell's relaxation treatment are described.

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Structural design requires a clear concept of what must be provided and this is often lost as analytical elaboration takes place. It is fitting then that the hundred pages of static analysis, as described, are followed by a chapter which crystallizes design approaches.

This is followed by two chapters which conclude the book and deal in rapid tempo with the vibration of suspension bridges. A considerable volume of modern work lies in this field and while Sir Alfred's closing chapters provide an excellent physical background and explanation of why failures have occurred, one wonders whether the condensation achieved here is not a little disproportionate. He does, however, provide references and gets across the ideas behind and the results of modern researches.

J. B. B. OWEN

Numerical Methods. By R. A. BUCKINGHAM. Pp. xii, 597. 70s. 1957. (Pitman)

The present book may well prove to be the last and definitive one on hand methods of numerical analysis. It is of a length which is seldom encountered nowadays and exhausts most of the classical possibilities of the subject. Dr. Buckingham derives some of his enthusiasm from the late Dr. Comrie, but has more recently concerned himself with the calculations involved in nuclear physics.

The first chapter forms an introduction to computation and follows well trodden paths, except that the use of differences for checking is introduced at an early point and before any real definition of the operation has been given. The treatment of errors and their effects is also rather fuller than usual, but this is compensated by the dismissal of automatic digital calculators in a page and a half.

Chapter 2, on sums and differences of polynomials, is also classical and leads to chapter 3 on Lagrangean interpolation and its consequences. In the fourth and fifth chapters a break with tradition is made when Newton's method of divided differences is used to derive the more useful central difference formulae. Curiously these chapters are *followed* by a short account of symbolic methods. An omission here is any detailed account of the Gaussian integration procedures which are now coming into prominence with the increasing use of computing machines.

In chapters 7 and 8 the solution of one point boundary condition differential equations is taken up, the treatment leans heavily on that of Milne and is unremarkable except for its omission of the Runge-Kutta methods.

Chapter 9 deals with non-linear algebraic equations and is interesting for the full account of Lin's method and of the Bernoulli-Aitken process, neither of which is readily available elsewhere.

Chapter 10 is concerned with aspects of the method of least-squares and ranges from the treatment of experimental data to Prony's method for exponential representation.

The next three chapters consider the problems involved in the solution of sets of linear simultaneous equations. All of the classical methods receive attention but the more recent descent methods are scarcely mentioned. In particular, the relaxation technique is described in a way which is unlikely to appeal to readers who have no previous acquaintance with the subject. A good point of treatment is the very full description of the evaluation of latent roots and vectors which appears in chapter 12.

Following these matters, chapter 14 applies some of the methods just developed to the solution of two point boundary condition differential equations and of Fredholm integral equations.

The book concludes with two chapters on two variable problems and on partial differential equations respectively, and with four appendices and a list of useful formulae.

The author has compiled a monumental work and has done a considerable service to teachers of the art by providing over 135 worked examples, mostly non-trivial. The prose style is clear and readable and the book can be unreservedly recommended.

A. D. BOOTH

Produits d'Inversions et Métrique Conforme. By RENÉ LAGRANGE. Cahiers Scientifiques XXIII. Pp. x, 331. 1957. (Gauthier-Villars, Paris)

This book is an exposition with amplification of six papers previously published in the Annales de l'École Normale Supérieure (1942, 1946), Acta Mathematica (1950, 1951), and in the Bulletin des Sciences Mathématiques (1950, 1951). It is written primarily for the professional mathematician, but any geometry enthusiast with a nodding acquaintance with the calculus of Pfaffian forms and exterior differentiation will derive both enjoyment and instruction from most of its pages, as the book is mainly self-contained and very clearly written.

The topic "products of inversions" has previously been studied by several mathematicians, but in the present work the approach is more geometrical. It is impossible in a brief review to do justice to the contents of this book, which should be regarded as a major work on a fairly specialized topic. By a "sphere" is meant an $(n-1)$-dimensional hypersphere in n-dimensional Euclidean space. It is evident that the process of inversion of elementary plane geometry admits an immediate generalization to such spheres, and this generalized inversion has the familiar property of mapping spheres into spheres, conserving angles, and mapping inverse points into inverse points. A product of inversions is obtained by successive inversions in two spheres, and the set of such products forms a group of point transformations called the anallagmatic group. This group can also be regarded as the group of point transformations which conserve spheres. The first part of the whole book is concerned essentially with properties associated with this group. The second chapter introduces the idea of product of points, spheres and planes and develops a suitable calculus. Further chapters consider anallagmatic transformations arising from successive inversions in n given spheres, and conditions when such products reduce to a similitude, a homothetic transformation, or an inversion.

The second part of the book is concerned with the anallagmatic metric and its consequences. The fundamental entities in anallagmatic geometry of E_n are hyperspheres of dimensions $n-p$ $(1 \leq p \leq n)$, the hypersphere of zero dimensions being a point pair. Since only the distances between two hyperspheres can be invariants of the geometry, and in particular because a point is not a hypersphere, it follows that the distance between a point and a hypersphere is a covariant and not an invariant. The properties associated with such a metric are investigated in the remaining 120 pages of the book.

The printing is excellent.

T. J. WILLMORE

Mechanik. By F. HUND. vii + 200 pp. 1956. (Teubner, Stuttgart)

This book is the first volume of an introductory treatise of three volumes on theoretical physics, and in it theoretical mechanics is so presented as to require of the reader a knowledge only of the elements of the calculus and of analytical geometry. In the preface, Professor Hund writes: "Wichtiger als

eine Kenntnis bestimmter mathematischer Methoden ist dabei die Bereitschaft des Lesers zu straffer Gedankenführung und die Fähigkeit, Anschauliches und Abstraktes miteinander zu verknüpfen." In these two respects, namely concerning the pure mathematical background and on the emphasis on physical sense, the book is like many English textbooks on the subject. In the latter, the reader is urged to do the exercises to get the feel of the subject; in this book there are no exercises but there are some examples in the text, and it is by them but more especially by an excellent selection of topics that the author illustrates the dynamical principles and inculcates the physical sense deemed so important. It is surprising how much is contained in this slim volume. The topics include damped, forced, and coupled oscillations, normal oscillations, the problem of two bodies, Newtonian relativity and Poinsot's representation of the motion of a rigid body. Some indication of the range of subjects can be derived from the chapter headings: Plane statics, Three-dimensional statics, Rectilinear motion, Motion of a particle in space, Moving frames of reference, Systems of particles, Motion of a particle on a surface, Motion of a rigid body, Analytical mechanics. There is also an appendix giving a brief historical survey. It is clear that a student who works through this book will not fail to gain a considerable knowledge of dynamics.

Vector notation is used throughout, and many fundamental results are proved by the vector methods, that is to say, by a diagram and an appeal to geometric intuition. Vector proofs have, of course, the advantages of economy and speed. The mathematical reader may think the economy and speed here is such that the mathematics tends to be squeezed out. He will certainly wish for a more adequate treatment of, say, angular velocity, tensors, the inertia tensor and the variational principles. A fuller treatment of such topics would, perhaps, require of the reader more mathematical knowledge than the author is prepared to ask, but one wonders, after all, whether the student who is to read dynamics as far as Lagrange's equations and Hamilton's principle is properly equipped if he does not know, say, the theory of matrices and the elements of linear algebra. With a not very extensive knowledge of these, theoretical mechanics can be presented in a way which could link up with modern theoretical physics.

Be that as it may, with the limitations which the author has set himself, he has produced a book with very few blemishes which provides a good down-to-earth account of theoretical mechanics.

E. V. WHITFIELD

Theory of Functions of a Complex Variable. By S. NARAYAN. Pp. 380. Rs. 12/8. 1956. (Chand, Delhi)

Mr. Narayan continues his series of texts for Indian university students with a sound, if somewhat uninspiring, book on complex variable, covering classical ground in familiar fashion. A good feature is the emphasis placed on the geometry of the subject; there is a long and useful chapter on the bilinear transformation $w = (a + bz)/(c + dz)$ with plenty of geometrical detail, and again a long and good chapter on the elementary conformal mappings, with details worked out in full and a generous supply of diagrams. The calculus of residues is treated fairly but not allowed to overshadow the rest of the book. On the other side, the last chapter, on uniform convergence, comes rather late and its motivation is not at all clear. The unfortunate student might well think that it is just something added on to make the whole subject harder; he would not easily realize that its whole purpose here is to ensure that $\lim f_n(z)$ is respectable.

The collection of examples for the reader is varied, adequate but not overpowering; there are practically no references for further reading.

T. A. A. BROADBENT

Topology. By E. M. Patterson. Pp. 128. 8s. 6d. 1956. (Oliver and Boyd, Edinburgh)

Elementary Topology. By D. W. Hall and G. L. Spencer. Pp. 303. 56s. 1955. (Wiley, New York; Chapman and Hall, London)

Dr. Patterson's little book is naturally only an introduction to the subject—no more could be attempted within the compass of the texts of this series—but is is remarkable how much the author covers in so little space. The writing is very clear and the subject matter very well arranged: I was particularly impressed by the chapters on abstract spaces. The book would make a good foundation for a first course in Topology and forms an excellent exposition for the professional mathematician who, while not specializing in Topology, would like to know what it is about.

There is one fallacy, namely in the second part of the proof of Theorem 3.15. Since Dr. Patterson includes some exercises for the reader, I should like to add one: detect the fallacy in 3.15 (3 marks) and give a correct proof (7 marks).

The American book is confined to "general point-set topology," the combinatory side of topology being excluded (except for the Jordan curve theorem). On the point-set side the book starts from the beginning and leads the reader by slow and careful steps to some very advanced points indeed. For these reasons the book can be recommended for a reader with a serious interest in the subject, particularly one who must in the main get along unaided. It assumes very little previous knowledge of mathematics, none at all of mathematical analysis: the properties of the continuum are derived from a very down-to-earth definition (implicit, not explicit) of a real number as a decimal. In the course of a proof at all long the authors are ever ready to halt and summarize the position reached, thus giving the reader a breather. As to the advanced points reached, they include Zippin's characterization of the topological 2-sphere, the theory of partitionable spaces and the relations between Zermelo's Axiom of Choice, Zorn's Lemma and the Tychonoff Product Theorem.

But there is to my mind a certain inconsistency in the book's make-up. The book's higher flights seem to me unsuitable, in degree of abstraction or of difficulty or of both, for a reader who has not already grasped the elements of mathematical analysis, or who cannot take rather more confident strides: one reader will find much of the book unnecessary or painfully slow, another will lose the wood in the trees. But this view may in part be due to my own old-fashioned ideas of priority in mathematics.

Within the domain of "point-set topology" the book clearly springs from the American school of R. L. Moore rather than from the European: this colours the choice of topics after the foundations have been laid.

Patterson's Theorem 3.15, that the product of two compact spaces is compact, is correctly proved here, on p. 84. But curiously enough the proof of Tychonoff's theorem, that the product of any number of compact spaces is compact, contains a rather similar slip.

H. D. Ursell

Éléments de Méchanique Quantique. By Ph. Pluvinage. Pp. 548. 4,000 Fr. 1955. (Masson et Cie, Paris)

This book is concerned with the principles of non-relativistic quantum theory and its applications in atomic and molecular physics.

It is written in the best French tradition, the exposition is very clear and the treatment analytic. Many of the textbooks at present extant follow a conventional pattern of development but in this book a more original line is followed. The emphasis is on principles, which are very carefully discussed, but at well-chosen stages illustrative applications are discussed which may be

taken from the physics of molecules, atoms, or nuclei. Another unusual feature is that states of the continuous spectrum are treated on the same footing as bound states—in most earlier texts at least, the latter are over-emphasized compared with the former.

The author has taken much trouble to develop classical theory in a form which is most appropriate for the introduction of quantum mechanics. This makes the introductory sections particularly stimulating and sane. The inclusion of a specially detailed treatment of the so-called WKB approximation is in keeping with the emphasis placed on relation to classical theory.

After an introductory chapter on the fundamental concepts of classical atomic physics the book is divided into three main sections—Linear Motion of a Particle (11 chapters), the Particle in Space (10 chapters) and Systems of Particles (5 chapters). It will be seen from the large number of chapters in the first main section that the author has chosen to expound the principles in relation to the simplest physically sensible conditions. In this way they are most clearly displayed.

There can be no hesitation in recommending this book to those who teach or use quantum mechanics,—the more so because of the different approach to the subject in the French style.

H. S. W. MASSEY

Das Lebesgue-Stieltjes Integral. By E. KAMKE. Pp. 226. D.M.20. 1956. (Teubner, Leipzig)

This is an entirely new rendering of an earlier well-known book by the author (*Das Lebesguesche Integral*, Leipzig-Berlin (Teubner) 1925). It now comprises general measure theory in n-dimensional space and the Lebesgue-Stieltjes integral for functions of n real variables. The earlier book dealt with one- and two-dimensional Lebesgue measure and with the Lebesgue integral for functions of one real variable. The geometrical approach is maintained, in which measure is considered as the basic notion and the integral as an application of measure to ordinate sets. The book begins with what is required from the theory of point sets and ends, as in the earlier edition, with an account of the Perron integral for functions of one variable.

Like all books of Kamke, this is distinguished by clarity and impeccable rigour. It attends, almost pedantically, to all details and leaves hardly anything to the reader's care. What makes this account particularly valuable is the fact that it does not restrict itself to the basic facts of the theory (as many modern books in their craving for brevity do); the reader will find in it everything that is likely to occur when using Stieltjes integrals. The book is intended as a text for German students. Alas, how few students in this country do ever really study this important subject. Here is their opportunity.

The reviewer has to make two criticisms. In the case of functions of one variable the measure-theoretical approach leads to the formula $\int_a^b d\varphi(x) = \varphi(b+0) - \varphi(a-0)$ while one would wish to have, as with the Riemann-Stieltjes integral, $\int_a^b d\varphi(x) = \varphi(b) - \varphi(a)$. This can, of course, be achieved by "truncating" the measuring function $\varphi(x)$, and the author should have indicated this. Again, his integral $\int_a^b f\,d\varphi$ is not additive in the sense that $\int_a^b = \int_a^c + \int_c^b$. This he could have achieved by normalizing $\varphi(x)$ to be continuous to the left; again some remark to this effect would have been helpful.

The book is published (and well printed) in East Germany. Was it therefore really necessary to relegate H. A. Schwarz to a footnote in honour of Bunjakowski? After all, it is really all Cauchy.

W. W. ROGOSINSKI

Theory of Functions I. II. By K. KNOPP. Translated by F. Bagemihl from the 5th German edition. Pp. vii, 146; x, 150. 11s. each part. (Mayflower Publishing Co., London; Dover Publications, New York)

Knopp's books on function theory were originally published in the "Sammlung Göschen"; they have now been translated and the whole set of five volumes is available, the others being an elementary, introductory volume and two problem books. The present two volumes start from the elements, but the novice would be well advised to acquaint himself with the account of complex numbers, simple mappings, and the elementary functions given in the introductory volume of the set. The main themes of classical 19th century function theory are developed rigorously, lucidly, and on the whole very readably, though the English of the translation is stilted. Part I is mainly the Cauchy theory—Cauchy's theorem, the Taylor and Laurent expansions, residues. Part II gives the Riemann-Weierstrass aspect—Weierstrass' factor theorem, Mittag-Leffler's theorem, elliptic functions, Riemann surfaces. To bring himself nearer the present day, the reader must go on to the newer books, for instance those by Carathéodory and Ahlfors; but for some time to come the classical theory is likely to remain an essential basis on which to build, and Knopp's account has the merits of accuracy, brevity, and clarity.

T. A. A. B.

Elements of Pure and Applied Mathematics. By H. LASS. Pp. xi, 491. $7.50 1957. (McGraw-Hill)

The author is a research specialist in jet propulsion at the California Institute of Technology, and the selection of topics in his book should therefore indicate the mathematical equipment needed by a high-grade technologist in an age of jets and atomic power; it is substantially free from the criticism often levelled against the "mathematics for engineers" text-book, that it deals with mathematics of 50 years ago by methods current 100 years ago.

It was, I believe, Professor Neville who remarked that a technologist does not need Professor Watson's knowledge of Bessel functions; he needs to know what Bessel functions are, and that everything worth knowing about them can be found in Watson's book. Dr. Lass has grasped this principle, and his book is not and is not meant to be a closely integrated, fully developed treatment. It falls into sections which are practically self-contained, and each section tries to bring into prominence the main principles; the details surrounding them are to be sought for in the worked and unworked exercises, and in the references given to more extensive accounts. Some care is taken to point out the sketchiness of certain arguments and the places where fuller information and strict proofs may be found.

Briefly, the included topics are: determinants, matrices, vectors and tensors; complex variable, including residues and the Schwarz-Christoffel mapping; differential equations, in both the real and the complex field, orthogonal polynomials, Fourier series and integrals; the Stieltjes integral and the Laplace transform; calculus of variations; group-theory; probability and the theory of games; real variable theory.

The treatment of vectors is thorough and some of us will be glad to see emphasis placed on ∇ itself rather than on its avatars of div, grad and curl. The definition of ∇ as

$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is perhaps the right one for the novice, but I wish that the author had in the later paragraphs laid more stress on the intrinsic (Hamiltonian) definition, by virtue of which ∇ is seen to be not just a convenient shorthand but an operator of immense power. Tensors are very neatly described, and their applications include Hamilton's equations and the Navier-Stokes equations; I am not sure, however, that a restriction to Cartesian tensors might not have allowed a more effective and equally useful exposition. Complex variable theory seems to sacrifice elegance to utility, but to compensate, the work on the nature and existence of solutions of differential equations is eminently clear and readable, and this commendation also applies to the sections on orthogonal polynomials. Of the remaining topics, groups and the Galois theory are surprising choices pleasantly expounded.

If space permitted, an interesting comparison could be made with the somewhat similar book by Murnaghan, *Introduction to applied mathematics*. Though Dr. Lass cannot provide the sudden stimulus, the flash of illumination, that the really bright student can get from Murnaghan again and again, he gives a steadier light and is often easier to read. A well-prepared mathematician working for an engineering degree would find this book valuable for its style, content and outlook value.

T. A. A. BROADBENT

Canon Arithmeticus von C. G. J. JACOBI, nach Berechnungen von W. PATZ in verbesserter und erweiterter Form neu herausgegeben von H. BRANDT. Pp. 16, 432. DM.46. 1956. (Akademie-Verlag, Berlin)

This volume illustrates the wisdom of the bibliographical rule that the title page is to be followed rather than the binding or the dust cover. Both the latter mention only Jacobi, and it is probably desirable not to omit this name in orders to booksellers; but the overlap with Jacobi's classic work of 1839 is limited. This is really the *Canon Arithmeticus* of the late Professor Brandt and Dr. Patz, of whom the former provided the introductory text and (one tends to gather) the original initiative, while the latter, already an experienced table-maker, computed the tables.

The core of both the old work and the new is a set of tables giving numbers (residues) x for given indices i, and *vice versa*, where x and i are connected by the binomial congruence

$$g^i - x \equiv 0 \pmod{p},$$

for each of the odd primes p less than 1000. But whereas Jacobi used as the base g that primitive root which seemed to him most convenient (for example 10, in the numerous cases in which 10 is a primitive root of p), Brandt and Patz invariably take g to be the *smallest* positive primitive root of p. Jacobi used this base for only 12 of the 167 primes in question; moreover, his tables contained rather numerous errors. Brandt and Patz provide, on the even pages from 2 to 348, tables lettered N giving the residue x as a function of i, and tables lettered I giving the index i as a function of x, arranged in much the same way as Jacobi's, but not usually exhibiting the same values because of the difference in bases referred to above. Facing these tables, on the odd pages from 3 to 349, are further tables of a kind not given by Jacobi, namely tables lettered I' giving the index of $(x + 1)$ as a function of the index of x, and tables lettered I'' giving the index of $(x - 1)$ as a function of the index of x. Brandt remarks that these are analogous to tables of addition and subtraction logarithms.

In a second part, on pages 353–426, Brandt and Patz give tables of types N and I for the 123 composite numbers P less than 1000 which are of the form p^m or $2p^m$, where p is an odd prime. The value of g is invariably the smallest positive primitive root of P. Jacobi gave tables, with the same base except

in one case ($P = 529$), for the 17 composites of the form p^m, but not for the 106 composites of the form $2p^m$.

Finally, in a small third part, on pages 429–432, Brandt and Patz give tables of types N and I for the powers of 2 from 2^4 to $2^{11} = 2048$, with $g = 3$ in each case. Jacobi gave such tables for the powers of 2 from 2^4 to $2^9 = 512$.

In the introductory matter, Brandt gives no fewer than 24 examples of the use of the tables in various types of problem in the theory of numbers. As the usefulness of Jacobi's canon is in any case generally recognized, it may confidently be expected that the version now offered, revised in form, extended in scope, freshly computed and (presumably) more accurate, will be found of great service in its field. The volume is well printed and produced.

A. FLETCHER

Linear Inequalities and Related Systems. Edited by H. W. KUHN and A. W. TUCKER. Pp. xxi, 322. $5.00. 1956. (Princeton University Press)

This book, No. 38 of Annals of Mathematics Studies, is a worthy successor to Nos. 24 and 28, Contributions to the Theory of Games, I and II, for which the same two editors were responsible. It contains 18 papers on subjects connected with Linear Programming and with Matrix Games. From an algebraic point of view (as distinct from that of numerical mathematics) Linear Programming is the theory of mixed systems of inequalities and equations, centred around the Fundamental Duality Theorem. One of its versions (which comprises the Main Theorem of game theory as a special case) is this:

Consider the two systems of inequalities (in matrix notation) $Ax \geq b$ and $y'A' \leq c$.

Then the minimum of $c'x$ equals the maximum of $y'b$.

Corollaries, lemmas and consequences related to this theorem are treated in four papers, written singly, and jointly, by Goldman and Tucker. It is shown, for instance, that a system $KW \geq 0$, $W \geq 0$, where K is skew symmetric, has a solution W^+ such that $KW^+ + W^+ > 0$ in all components.

Ky Fan discusses similar problems in greater generality and abstractness, while Duffin deals with analogous problems in infinite dimensional spaces.

A paper by Dantzig, Ford and Fulkerson contains a new routine for the numerical solution of Linear Programming problems, and two papers, by Mills and Wolfe, deal with matrix games. There follow six papers on combinatorial problems. They concern, essentially, systems of linear equations where only solutions in integral values are of interest and point to a most interesting development of linear programming. Earlier theorems by P. Hall and by R. Radó belong to this field. The authors of these papers are Dantzig, Fulkerson, Gale, Heller, Hoffman, Kruskal, Kuhn, and Tompkins in various combinations. Finally, three papers by Kuhn, Thompson, and Gale deal with models of production, as initiated by Wald and von Neumann.

The book ends with a bibliography of 289 titles.

As all these studies, this one is also written mainly for experts in the several fields that are covered. But they are also extremely useful as reference books, and as indications of the trend of development. As in the case of the two predecessors mentioned above, the editors contributed an excellent Preface. It paraphrases the papers in a way which makes them much easier to appreciate, even to a newcomer to the subject.

We have every reason for looking forward to Annals of Mathematics Studies 39, which will be volume III of the Contributions to the Theory of Games.

S. VAJDA

Memorial des Sciences Mathématiques—Part CXXXIV. **Les méthodes d'approximation variationelles dans la théorie des collisions atomiques et dans la physique des piles nucléaires.** By T. KAHAN, G. REDEAU, and P. ROUSSOPOULOS. Pp. 82. 1956. (Gauthier-Villars, Paris)

One of the developments hastened by military applications in the last World War is that of the application of variational methods to the quantitative study of states of the continuous spectrum of mechanical, electrical, atomic and nuclear systems.

This has meant that many-body problems may be treated in a manner not hitherto possible. On the other hand the method is not so satisfactory as for the investigation of bound states. In the latter case a definite upper bound to the energy of the state is obtained but in the application to open or free states the corresponding quantity, which may be a phase shift, may be either over- or under-estimated. So far there has been little development of methods of discrimination and considerable care must be taken to ensure that the variational procedure gives sensible results.

For these reasons it is particularly important that the basis of the procedure should be fully understood. The authors of this tract have made a useful contribution in this respect. There is first an introductory chapter on the wave mechanical theory of collisions, followed in the next two chapters by a general discussion of the variational formulation of collision problems, including multiple scattering. Chapter IV discusses actual applications to atomic collision theory and to the scattering of electromagnetic or sound waves by obstacles. The final chapter is concerned with applications to problems of diffusion, with particular reference to that of neutrons in a reactor.

The style is clear and the material interesting though it would have seemed worthwhile for the authors to attempt to compare and contrast the many variational procedures which have been suggested, in terms of their general formulation. The tract should be useful to those working in any of the fields to which the theory is applicable.

H. S. W. MASSEY

Mass und Integral und Ihre Algebraisierung. By C. CARATHÉODORY. Pp. 337. D.M.38.50. 1956. (Birkhauser, Basel and Stuttgart)

Among the great contributions of the last Constantin Caratheódory to numerous branches of both Pure and Applied Mathematics, his work on the theory of measure, which forms the basis of the majority of modern expositions of measure in abstract spaces, were notable. The present book aims at a further generalization of this theory which is prompted, among other things, by the following considerations. Modern theories of measure and area are theories of the measures of sets of points. In contrast to this, the ancient theory of Euclid is a theory of measure of geometrical configurations: that there is a difference is shown, for example, in that in adding two rectangles to form a third the common side is neglected. One might explain this discrepancy by saying that in fact the Riemann or Lebesgue theories do not deal with the measures of sets of points, but with the measures of classes of sets which differ only by sets of measure zero. The aim of Carathéodory's theory is to set out the theory of measure for the most general objects to which it can be applied.

The theory which is developed is that known in English and French speaking countries as the theory of measure on the elements of a Boolean ring. Carathéodory develops this theory on the basis of a set of axioms concerning the elements of the ring—which he calls soma—in which the fundamental relationship is that of disjointness of two elements: and he also expounds and compares with this the equivalent set of axioms based on the operations of

union and intersection in a Boolean ring. A considerable, and fairly complicated, development is needed for the theory of real valued functions on such rings. In the classical theory, the functions are defined by assigning to each point of a set the values corresponding to it; this method is not available, and must be replaced by one in which to each real number is assigned an element of the ring—a construction parallel to Lebesgue's decomposition of a set according to the values of the function assumed. The technique of calculation with these "Ortsfunktionen" takes up chapters 3 and 4, the first two being concerned with the definition of the "soma." Following on this is a chapter on measure functions, in which measurability is defined in a manner parallel to that for abstract sets of points, and then one on the definition of the integral of "Ortsfunktionen." The last five chapters of the book develop the theory of measure functions and of integrals further, proving for the very general case envisaged, theorems on inner and outer measure, the Jordan decomposition, the ergodic theorem, Vitali covering theorem, and so on. This part of the book forms a very good and searching account of all those parts of the theory of measure and integration which do not require the measure to be one defined on a space with a topological structure. An appendix discusses other axiom systems for the rings of "soma," and also such interesting questions as the possibility of such a ring being represented by a ring of sets.

This book is a posthumous work, having been edited by Professors Finsler, Rosenthal, and Steuerwald. The work has been excellently done: the only criticism one could make is that it is a pity that there is so little reference to the considerable work by American authors on the theory of measure in Boolean rings. The press has also done its work excellently; and the result can be recommended to all interested in measure theory. J. L. B. Cooper

Darstellende Geometrie. By H. von Sanden. Pp. 115/40. D.M.7.80. 1956. (Teubner, Leipzig)

In the preface the author states that in writing this book his aim has been to present "Darstellende Geometrie" as a means of clearly representing spatial conceptions and not merely as an introduction to projective geometry, and a careful study of the book suggests that he has achieved that aim. The use of analytical geometry, where appropriate, simplifies much of the explanatory text: it also indicates to the mathematician how his subject is put to practical use. The book, which contains no examples for solution by the reader, is not so much a text-book as a very detailed set of lecture notes on a large number of topics.

In the first chapter the reader is introduced to the two-dimensional representation of points, lines and planes in space, to the principle of rabattment and its applications. Construction of the ellipse as a projection of a circle, as a section of a cylinder and by the trammel method, together with properties of conjugate diameters and the circles of curvature at the extremities of the major and minor axes, is dealt with in the second chapter. Then follow chapters on sections of a cone, surfaces of revolution, pipes (including curves of intersection) the helix and helical surfaces, axonometry and perspective. An interesting feature is that all the diagrams are collected into a small loose booklet, which is placed inside the back cover of the book. This avoids the necessity for turning pages backward and forward, which is invariably the case when a diagram is printed as an inset to a long explanatory text.

The book is very well printed on a pleasantly matt surface paper, and the diagrams are beautifully clear and easy to follow. On the whole the style is direct and presents no great difficulty, but there is the occasional sentence written in a somewhat severe style which seems to present difficulty—even to the student whose native tongue is German.

Apart from this small adverse criticism the book may well be recommended to mathematicians responsible for introducing potential engineers and architects to the field of geometrical drawing.

F. T. CHAFFER

Numerical Analysis: Proceedings of Symposia in Applied Mathematics, Vol. VI. Edited by JOHN H. CURTISS. Pp. 303. $9.75. (1956). McGraw-Hill, New York

A very wide range of subjects is covered by these papers, which also vary in their character; some might be regarded as statements of progress in a particular field while others make considerable contributions to research. Papers are included on such widely differing topics as partial differential equations, gas dynamics, integral equations, combinatorial problems, number theory, conformal mapping, the theory of approximations and the method of steepest descents, while the mathematics arising from problems in planning is well represented by articles on dynamic programming and the assignment problem.

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